# ON CALABI–YAU THREEFOLDS ASSOCIATED TO A WEB OF QUADRICS

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Abstract. We study the geometry of the birational map between an intersection of a web of quadrics in $\mathbb{P}_7$ that contains a plane and the double octic branched along the discriminant of the web.

## Introduction

It is a classical fact that there is a correspondence between the base locus $S$ of a net of quadrics in $\mathbb{P}_5$ and the double sextic branched along the discriminant of the net. The latter is the moduli space of certain rank-2 sheaves on the former (see [26]). Moreover, if the base locus contains a line $L$, then the two surfaces are birational. More general conditions for the existence of a birational map were given by Nikulin and Madonna (see [22] and its sequels).

A precise description of the birational map between the surface $S$ and the double sextic can be found in [7]. In this case, $S$ is the blow-up of the double sextic along rank-4 quadrics in the net. The latter results from the fact that the map defined by the linear system $|2H - 3L - \sum_1^k L_i|$, where $H$ is the hyperplane section in $\mathbb{P}_5$ and $L_i$ are the lines on $S$ that meet $L$ (see [7, Thm 3.3]), is hyperelliptic. Moreover, one can show that the birational map factors through another K3 surface (a space quartic that contains a twisted cubic) and its geometry (e.g. the contracted curves) is governed by the behaviour of the lines $L_i$. The birational map between the two surfaces can be also constructed via an incidence variety ([18]). The latter construction was adopted in [24] to the case of a generic web $W = \mathrm{span}(Q_0, Q_1, Q_2, Q_3)$ in $\mathcal{O}_{\mathbb{P}_7}(2)$, such that its base locus $X_{16}$ contains a fixed plane $\Pi$. More precisely, using Bertini-type and computer algebra arguments, Michałek proved that if we put $S_8$ (resp. $X_8$) to denote the discriminant surface of the web $W$ (resp. the double cover of the web $W$ branched along the discriminant surface $S_8$) and $W$ is generic enough, then the Calabi-Yau varieties $X_{16}$ and $X_8$ are birational. However, the approach of [24] gives neither explicit sufficient condition for birationality of $X_{16}$ and $X_8$ nor a method to study the geometry of the map.

In this paper, for the matrices $\mathfrak{q}_0, \ldots, \mathfrak{q}_3$ that give the quadrics $Q_0, \ldots, Q_3 \in \mathcal{O}_{\mathbb{P}_7}(2)$ such that $Q_0 \cap \ldots \cap Q_3$ contains a plane $\Pi$ we define two auxiliary matrices $\mathfrak{a}$, $\mathfrak{A}$ and use them to obtain a surface $\mathcal{B} \subset \mathbb{P}_4$ and a three-dimensional quintic $X_5 \subset \mathbb{P}_4$ that contains the surface $\mathcal{B}$. Then, under the assumptions

**[A1]:** $X_{16}$ has exactly 10 singularities on $\Pi$ and is smooth away from the plane $\Pi$,
**[A2]:** no 4 singular points of $X_{16}$ lie on a line,
**[A3]:** the set $\{\underline{x} \in \mathcal{B} : \mathrm{rank}(\mathfrak{A}(\underline{x})) \leq 2\}$ consists of 46 points ,
**[A4]:** the discriminant surface $S_8$ has only isolated singularities,

we show that there is a birational map $X_{16} \dashrightarrow X_8$ that factors as the composition

$$X_{16} \overset{\sigma^{-1}}{\dashrightarrow} \tilde{X}_{16} \xrightarrow{\pi} X_5 \overset{\psi^{-1}}{\dashrightarrow} \tilde{X}_5 \xrightarrow{\hat{\phi}} X_8\,,$$

where $\sigma, \psi$ are certain blow-ups, $\pi$ is resolution of the projection from $\Pi$ and $\hat{\phi}$ is obtained via Stein factorization from restriction of the so-called Bordiga conic bundle to the blow-up of the quintic

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$X_5$. In particular, under the above assumptions $\mathcal{B}$ is the so-called (smooth) Bordiga sextic. Bordiga sextic and Bordiga conic bundle have been studied already by the Italian school (see [30], [2] and the bibliography in the latter), so the above factorization enables us to give a precise description of the geometry of the birational map in question. In particular, we are able to show that the map has no two-dimensional fibers, describe the contracted curves (Thm 3.6), classify the singularities of the discriminant of the web (and prove that all of them admit a small resolution) and give an upper bound of their number (see Cor. 4.7).

Our considerations yield that the assumptions [A1],...,[A4] are fulfilled by a generic web of quadrics such that its base locus contains a fixed plane. Careful analysis of our arguments shows that one can assume less in order to obtain a birational map $X_{16} \dashrightarrow X_8$, but once one omits the above assumptions the geometry of the birational map changes. For instance, if [A2] is not satisfied, the surface in $\mathbb{P}_4$ one obtains as a result of the projection is no longer the Bordiga surface, without [A1] (resp. [A3]) the threefold $X_{16}$ (resp. $X_5$) has higher singularities etc. Still, the main strategy we use can be applied to study those degenerations - we do not follow this path in order to maintain the paper compact.

Our motivation is twofold. First, it seems a natural question to ask under what assumptions a three-dimensional Calabi-Yau analogue of the well-known result on K3 surfaces holds. Second, we obtain a very precise description of a map between certain Calabi-Yau manifolds that (with help of a computer algebra system applied to a given example) could be of interest on its own, for instance as a source of examples of small resolutions.

The paper is organized as follows. In Sect. 1 we study the singularities of the threefold $X_{16}$ and Hodge numbers of its blow-up $\tilde{X}_{16}$. Sect. 2 is devoted to properties of projection from the plane $\Pi$. In the next section we describe the behaviour of the restriction of Bordiga conic bundle to the blow-up of the quintic $X_5$ we defined in Sect. 2. Finally, the last part (Sect. 4) contains a classification of singularities of the discriminant of the web and proof of main results of the paper.
*Convention:* In this note we work over the base field $\mathbb{C}$. By an abuse of notation we use the same symbol to denote a homogeneous polynomial and its zero–set in projective space.

## 1. Singularities of the intersection of four quadrics and a small resolution

Let $Q_0, Q_1, Q_2, Q_3 \subset \mathbb{P}_7$ be linearly independent quadrics that contain a (fixed) plane $\Pi$ and let

$$X_{16} := Q_0 \cap Q_1 \cap Q_2 \cap Q_3$$

be their (scheme-theoretic) intersection.

Without loss of generality we can assume that $\Pi := \{(x_0 : \ldots : x_7) \, : \, x_0 = \ldots = x_4 = 0\}$, which implies that each $Q_i$ is given by the matrix

$$\mathfrak{q}_i = \left[ \begin{array}{c|ccc} \underline{\mathfrak{q}}_i & & \mathfrak{b}_i^T & \\ \hline & 0 & 0 & 0 \\ \mathfrak{b}_i & 0 & 0 & 0 \\ & 0 & 0 & 0 \end{array} \right],$$

where $\underline{\mathfrak{q}_i}$ is a $5 \times 5$ matrix, $\mathfrak{b}_i := \begin{bmatrix} \mathfrak{l}_i \\ \mathfrak{m}_i \\ \mathfrak{n}_i \end{bmatrix}$ and $\mathfrak{l}_i, \mathfrak{m}_i, \mathfrak{n}_i \in \mathbb{C}^5$ are row–vectors. Moreover, in order to simplify our notation we put $\mathfrak{b}(y) := \sum_i y_i \mathfrak{b}_i$ and

$$\mathfrak{c}(x_5, x_6, x_7) := x_5 \begin{bmatrix} \mathfrak{l}_0^T & \mathfrak{l}_1^T & \mathfrak{l}_2^T & \mathfrak{l}_3^T \end{bmatrix} + x_6 \begin{bmatrix} \mathfrak{m}_0^T & \mathfrak{m}_1^T & \mathfrak{m}_2^T & \mathfrak{m}_3^T \end{bmatrix} + x_7 \begin{bmatrix} \mathfrak{n}_0^T & \mathfrak{n}_1^T & \mathfrak{n}_2^T & \mathfrak{n}_3^T \end{bmatrix} .$$

We have (compare [24, Prop. 1.8])

**Lemma 1.1.**

$$\mathrm{sing}(X_{16}) \cap \Pi = \{(0 : \ldots : x_5 : x_6 : x_7) \; : \; \mathrm{rank}(\mathfrak{c}(x_5, x_6, x_7)) \leq 3\}$$

*In particular, if the set* $\mathrm{sing}(X_{16}) \cap \Pi$ *is finite, then it consists of at most* 10 *points.*

*Proof.* Observe that the intersection $X_{16}$ is singular at a point $x$, iff the differentials $dQ_i(x) = (\mathfrak{q}_i x)^T$ of quadratic forms $Q_i$ at $x$ are linearly dependent, that is if there exists $(y_0 : \cdots : y_3) \in \mathbb{P}_3$ such that

$$\sum_{i=0}^{3} y_i \mathfrak{q}_i x = 0.$$

For $x = (0 : \cdots : 0 : x_5 : x_6 : x_7) \in \Pi$ the above condition reduces to $\sum y_i(x_5 \mathfrak{l}_i^T + x_6 \mathfrak{m}_i^T + x_7 \mathfrak{n}_i^T) = 0$. We can rewrite the latter as

$$\mathfrak{b}(y)^T (x_5, x_6, x_7)^T = 0. \tag{1}$$

For a fixed $y \in \mathbb{P}_3$ there exists a point in $\Pi$ satisfying the above relation iff $\mathrm{rank}(\mathfrak{b}(y)) \leq 2$. Moreover, for every $(x_5, x_6, x_7)$ and $y$ we have

$$\mathfrak{c}(x_5, x_6, x_7) y = \mathfrak{b}(y)^T (x_5, x_6, x_7)^T . \tag{2}$$

Therefore, $(0, \ldots, 0, x_5, x_6, x_7)$ is a singularity of $X_{16}$ iff there exist $y \in \mathbb{P}_3$ such that $\mathfrak{c}(x_5, x_6, x_7) y = 0$ or equivalently

$$\mathrm{rank}(\mathfrak{c}(x_5, x_6, x_7)) \leq 3.$$

Finally, suppose that the set $\mathrm{sing}(X_{16}) \cap \Pi$ is finite. Then, the number of its elements does not exceed the degree of the determinantal variety of $4 \times 5$ matrices of rank $\leq 3$. The latter is 10 by [14, Ex. 14.4.14] (see also [19], [27]). ☐

From now on we make the following **assumption**:

**[A1]:** $X_{16}$ *has exactly* 10 *singularities on* $\Pi$ *and is smooth away from the plane* $\Pi$,

As an immediate consequence of [A1] we obtain

*Remark* 1.2. For each $y \in \mathbb{P}_3$ we have $\mathrm{rank}(\mathfrak{b}(y)) \geq 2$. Indeed, we assumed that $X_{16}$ has only isolated singularities on $\Pi$. Therefore, for a fixed $y \in \mathbb{P}_3$, there exists at most one point in $\Pi$ satisfying the relation (1), so $\mathrm{rank}(\mathfrak{b}(y))$ cannot be lower than 2.

Lemma 1.1 and [6] support the following conjecture.

*Conjecture* 1.3. a) A nodal complete intersection of four quadrics in $\mathbb{P}_7$ with at most nine nodes is $\mathbb{Q}$-factorial.
b) A nodal complete intersection of four quadrics in $\mathbb{P}_7$ with exactly ten nodes that is not $\mathbb{Q}$-factorial contains a plane $\Pi$.

**Lemma 1.4.** *Suppose that [A1] holds.*
*a) The ideal of the set* $\mathrm{sing}(X_{16}) \cap \Pi$ *is generated by all* $4 \times 4$ *minors of the matrix* $\mathfrak{c}(x_5, x_6, x_7)$*. In particular, the ideal in question contains no cubics.*
*b) For each* $x \in sing(X_{16})$ *there exists precisely one quadric in* $W$ *such that* $x$ *is its singularity.*
*c) There exist three quadrics in the web* $W$ *that meet transversally.*
*d) The set* $\{y \in \mathbb{P}_3 \, : \, \mathrm{rank}(\mathfrak{b}(y)) = 2\}$ *consists of precisely* 10 *points.*

*Proof.* a) Recall that the determinantal variety $\mathbb{P}(\mathcal{V}_{10}) \subset \mathbb{P}_{19}$ given by the condition

$$\mathrm{rank} \begin{bmatrix} z_0 & \dots & z_4 \\ \vdots & & \vdots \\ z_{15} & \dots & z_{19} \end{bmatrix} \leq 3$$

has dimension 17 and degree 10. Moreover, the ideal generated by $4 \times 4$ minors of the above matrix is perfect by [12] (see also [5, Cor. 2.8]). Therefore, the ring $\mathbb{C}[z_0, \dots, z_{19}]/_{\mathrm{I}(\mathcal{V}_{10})}$ is Cohen-Macaulay. The map $(x_5, x_6, x_7) \mapsto \mathfrak{c}(x_5, x_6, x_7)$ parametrizes a 3-plane $\mathcal{P} \subset \mathbb{C}^{20}$ that meets $\mathcal{V}_{10}$ along ten lines. Since the ideal $\mathrm{I}(\mathcal{P})$ in the ring $\mathbb{C}[z_0, \dots, z_{19}]/_{\mathrm{I}(\mathcal{V}_{10})}$ is generated by 17 linear forms, it satisfies the assumptions of [13, Prop. 18.13]. Consequently, the quotient $\mathbb{C}[z_0, \dots, z_{19}]/(\mathrm{I}(\mathcal{V}_{10}) + \mathrm{I}(\mathcal{P}))$ is 1-dimensional Cohen-Macaulay and the ideal $\mathrm{I}(\mathcal{V}_{10}) + \mathrm{I}(\mathcal{P})$ coincides with its radical.
b) The plane $\mathbb{P}(\mathcal{P}) \subset \mathbb{P}_{19}$ meets the variety $\mathbb{P}(\mathcal{V}_{10})$ in exactly ten points, so none of the latter belongs to $\mathrm{sing}(\mathbb{P}(\mathcal{V}_{10}))$. But, as one can check by direct computation (see also [30]), all points of $\mathcal{V}_{10}$ that satisfy the condition

$$\mathrm{rank} \begin{bmatrix} z_0 & \dots & z_4 \\ \vdots & & \vdots \\ z_{15} & \dots & z_{19} \end{bmatrix} \leq 2$$

are its singularities. The latter implies that

$$\forall_{x \in \mathrm{sing}(X_{16})} \quad \mathrm{rank}(\mathfrak{c}(x_5, x_6, x_7)) = 3 \,. \tag{3}$$

Consequently, there exists precisely one $y \in \mathbb{P}_3$ that lies in the kernel of the matrix $\mathfrak{c}(x_5, x_6, x_7)$. By (2), the latter is equivalent to the condition $(0 : \ldots : x_5 : x_6 : x_7) \in \mathrm{sing}(Q(y))$. In this way we have shown the claim b).
c) follows from b) by standard arguments.
d) Suppose that a point $y \in \mathbb{P}_3$ satisfies the relation (1) for two various points in $\Pi$. Then, the line spanned by both points in question lies in the kernel of the matrix $\mathfrak{b}(y)$ and $\mathrm{rank}(\mathfrak{b}(y)) < 2$, which is impossible by Remark 1.2. In this way we have shown that

$$\#\{y \in \mathbb{P}_3 \, : \, \mathrm{rank}(\mathfrak{b}(y)) = 2\} \geq \# \, \mathrm{sing}(X_{16}).$$

The other inequality has been shown in the proof of part b). □

**Lemma 1.5.** *Assume that* $Z_P = \{f(y_1, \dots, y_4) = 0\} \subset \mathbb{C}^4$ *is a three-dimensional isolated hypersurface singularity that contains the germ of the plane* $\{y_1 = y_2 = 0\}$*. If the ideal*

$$\langle \frac{\partial f}{\partial y_1}, \dots, \frac{\partial f}{\partial y_4}, f, y_1, y_2 \rangle \subset \mathcal{O}_{\mathbb{C}^4, P}$$

*is maximal, then* $Z_P$ *is a node.*

*Proof.* We are to show that hessian of $f$ in $P$ does not vanish. Let $f_1, f_2 \in O_{\mathbb{C}^4, P}$ satisfy the condition $f = y_1 \cdot f_1 + y_2 \cdot f_2$. By direct computation we have

$$\langle f_1, f_2, y_1, y_2 \rangle = \langle y_1, y_2, y_3, y_4 \rangle \,. \tag{4}$$

Consider the linear parts $f_i^{(1)} = \sum_{j=1}^4 f_{i,j}^{(1)} y_j$ for $i = 1, 2$. Then hessian of $f$ in $P$ is given by

$$\det \begin{bmatrix} f_{1,1}^{(1)} & \frac{f_{1,2}^{(1)}+f_{2,1}^{(1)}}{2} & \frac{f_{1,3}^{(1)}}{2} & \frac{f_{1,4}^{(1)}}{2} \\ \frac{f_{1,2}^{(1)}+f_{2,1}^{(1)}}{2} & f_{2,2}^{(1)} & \frac{f_{2,3}^{(1)}}{2} & \frac{f_{2,4}^{(1)}}{2} \\ \frac{f_{1,3}^{(1)}}{2} & \frac{f_{2,3}^{(1)}}{2} & 0 & 0 \\ \frac{f_{1,4}^{(1)}}{2} & \frac{f_{2,4}^{(1)}}{2} & 0 & 0 \end{bmatrix} = -\det \begin{bmatrix} \frac{f_{1,3}^{(1)}}{2} & \frac{f_{2,3}^{(1)}}{2} \\ \frac{f_{1,4}^{(1)}}{2} & \frac{f_{2,4}^{(1)}}{2} \end{bmatrix}^2 .$$

To show that the right-hand side of the latter equality does not vanish put $y_1 = y_2 = 0$ in (4). $\square$

**Lemma 1.6.** *If [A1] holds, then all singularities of $X_{16}$ are nodes (i.e. $A_1$ points).*

*Proof.* Without loss of generality we can assume that all singularities of $X_{16}$ lie in the affine chart $x_7 \neq 0$ and the variety $Y := Q_0 \cap Q_1 \cap Q_2$ is smooth (see Lemma 1.4). By abuse of notation we use the same symbol to denote a quadric and the dehomogenization of its equation (i.e. $x_7 = 1$).

Observe that putting $x_0 = x_1 = \cdots = x_4 = 0$ in the ideal $\langle \bigwedge^4 \mathrm{Jac}(Q_0, \ldots, Q_3), Q_0, \ldots, Q_3 \rangle$ we get the ideal in $\mathbb{C}[x_5, x_6]$ generated by $4 \times 4$ minors of the matrix $\mathfrak{c}(x_5, x_6, 1)$. In particular, (see Lemma 1.1) we can compute the dimension of the $\mathbb{C}$-vector space

$$\dim(\mathbb{C}[x_0, \ldots, x_6]/_{\langle \bigwedge^4 \mathrm{Jac}(Q_0,\ldots,Q_3), Q_0,\ldots,Q_3, x_0,\ldots,x_4 \rangle}) = 10.$$

Moreover, the assumption [A1] yields an isomorphism

$$\bigoplus_{P \in \mathrm{sing}(X_{16})} \mathcal{O}_{\mathbb{C}^7,P}/_{\langle \bigwedge^4 \mathrm{Jac}(Q_0,\ldots,Q_3), Q_0,\ldots,Q_3, x_0,\ldots,x_4 \rangle \mathcal{O}_{\mathbb{C}^7,P}} \simeq \mathbb{C}[x_0, \ldots, x_6]/_{\langle \bigwedge^4 \mathrm{Jac}(Q_0,\ldots,Q_3), Q_0,\ldots,Q_3, x_0,\ldots,x_4 \rangle}$$

Therefore, for each $P \in \mathrm{sing}(X_{16})$, we have

$$\dim(\mathcal{O}_{\mathbb{C}^7,P}/_{\langle \bigwedge^4 \mathrm{Jac}(Q_0,\ldots,Q_3), Q_0,\ldots,Q_3, x_0,\ldots,x_4 \rangle \mathcal{O}_{\mathbb{C}^7,P}}) = 1 . \tag{5}$$

Fix a point $P \in \mathrm{sing}(X_{16})$ and assume that the germ of $Y$ near $P$ can be (analytically) parametrized as the graph of a map $(x_4(x_0, \ldots, x_3), \ldots, x_6(x_0, \ldots, x_3))$. Let $\tilde{Q}_3$ be the composition of the above parametrization with (the dehomogenized equation of) the quadric $Q_3$. By direct computation, (5) implies that the ideal

$$\langle \tilde{Q}_3, \frac{\partial \tilde{Q}_3}{\partial x_0}, \ldots, \frac{\partial \tilde{Q}_3}{\partial x_3} \rangle + \mathrm{I}(\Pi) \subset \mathcal{O}_{Y,P}$$

is maximal. By Lemma 1.5 the point $P$ is an $\mathrm{A}_1$ singularity of $X_{16}$. $\square$

We introduce the following notation:

$$\sigma : \tilde{X}_{16} \to X_{16} \tag{6}$$

is the blow-up of $X_{16}$ along the plane $\Pi$ and $S$ stands for the *strict transform* of the plane $\Pi$ under the blow-up $\sigma$. The variety $\tilde{X}_{16}$ is smooth and the blow-up in question replaces the 10 nodes with 10 disjoint smooth rational curves

$$E_1, \ldots, E_{10} \subset S. \tag{7}$$

**Convention:** *In the sequel, we shall identify smooth points of $X_{16}$ with their images in $\tilde{X}_{16}$, i.e. write $P$ instead of $\sigma(P)$ whenever it leads to no ambiguity.*

In the next section we will use the following lemma.

**Lemma 1.7.** *The variety $\tilde{X}_{16}$ is a projective Calabi–Yau manifold with the following Hodge diamond*

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 2 & & 0 & \\ 1 & & 56 & & 56 & & 1 \\ & 0 & & 2 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

*Proof.* By Lemma 1.4.b we can assume that $Y = Q_0 \cap Q_1 \cap Q_2$ is smooth. Let $\sigma : \tilde{Y} \longrightarrow Y$ be the blow–up of $Y$ along $\Pi$ with exceptional divisor $E$. We have

$$\begin{aligned} &\sigma_* \mathcal{O}_{\tilde{Y}}(kE) = \mathcal{O}_Y, \text{ for } k \geq 0, \\ &R^1\sigma_* \mathcal{O}_{\tilde{Y}}(E) = 0, \\ &R^1\sigma_* \mathcal{O}_{\tilde{Y}}(2E) = \mathcal{O}_\Pi(-1). \end{aligned}$$

Since $\mathcal{O}_{\tilde{Y}}(\tilde{X}_{16}) = \sigma^*\mathcal{O}_Y(X) \otimes \mathcal{O}_{\tilde{Y}}(-E)$ using the projection formula we get

$$\begin{aligned} &\sigma_* \mathcal{O}_{\tilde{Y}}(-k\tilde{X}_{16}) = \mathcal{O}_Y(-kX), \text{ for } k \geq 0, \\ &R^1\sigma_* \mathcal{O}_{\tilde{Y}}(-\tilde{X}_{16}) = 0, \\ &R^1\sigma_* \mathcal{O}_{\tilde{Y}}(-2\tilde{X}_{16}) = \mathcal{O}_\Pi(-5). \end{aligned}$$

The Leray spectral sequence and the Kodaira vanishing imply

$$H^i(\mathcal{O}_{\tilde{Y}}(-\tilde{X}_{16})) = 0 \text{ for } i \leq 3, \quad H^4(\mathcal{O}_{\tilde{Y}}(-\tilde{X}_{16})) \cong \mathbb{C}.$$

Since

$$\begin{aligned} &H^i(\mathcal{O}_Y(-2Y)) = 0, \text{ for } i \leq 3, \\ &H^4(\mathcal{O}_Y(-2X)) \cong H^0(\mathcal{O}_Y(2)) \cong \mathbb{C}^{33}, \\ &H^4(\mathcal{O}_{\tilde{Y}}(-2\tilde{X}_{16})) \cong H^0(\mathcal{O}_{\tilde{Y}}(\tilde{X}_{16})) \cong H^0(\mathcal{O}_Y(X) \otimes I(\Pi)) \cong \mathbb{C}^{27}, \\ &H^i(R^1\sigma_*(\mathcal{O}_{\tilde{Y}}(-2\tilde{X}_{16}))) = 0, \text{ for } i = 0, 1 \\ &H^2(R^1\sigma_*(\mathcal{O}_{\tilde{Y}}(-2\tilde{X}_{16}))) \cong H^2(\mathcal{O}_\Pi(-5)) \cong \mathbb{C}^6 \end{aligned}$$

the Leray spectral sequence implies

$$H^i(\mathcal{O}_{\tilde{Y}}(-2\tilde{X}_{16})) = 0, \text{ for } i \leq 3$$

and consequently

$$H^i(\mathcal{N}^\vee_{\tilde{X}_{16}|\tilde{Y}}) = 0 \text{ for } i \leq 2.$$

From the exact sequence

$$0 \longrightarrow \sigma^*\Omega^1_Y \longrightarrow \Omega^1_{\tilde{Y}} \longrightarrow \Omega^1_{E/\Pi} \longrightarrow 0$$

we get

$$\sigma_*\Omega^1_{\tilde{Y}} = \Omega^1_Y, \quad R^1\sigma_*\Omega^1_{\tilde{Y}} = \mathcal{O}_\Pi$$

and so

$$H^1\Omega^1_{\tilde{Y}} \cong \mathbb{C}^2.$$

Similarly, the exact sequence

$$0 \longrightarrow \sigma^*(\Omega^1_Y(-X)) \otimes \mathcal{O}_{\tilde{Y}}(E) \longrightarrow \Omega^1_{\tilde{Y}}(-\tilde{X}_{16}) \longrightarrow \Omega^1_{E/\Pi}(-1) \otimes \sigma^*\mathcal{O}_Y(-X) \longrightarrow 0$$

implies

$$\sigma_*\Omega^1_{\tilde{Y}}(-\tilde{X}_{16}) \cong \Omega^1_Y(-X) \text{ and } R^1\sigma_*\Omega^1_{\tilde{Y}}(-\tilde{X}_{16}) \cong \mathcal{N}_{\Pi|Y} \otimes \mathcal{O}_Y(-X).$$

Twisting the exact sequence

$$0 \longrightarrow \mathcal{N}_{\Pi|Y} \longrightarrow \mathcal{N}_{\Pi|\mathbb{P}^7} \longrightarrow \mathcal{N}_{Y|\mathbb{P}^7}|\Pi \longrightarrow 0$$

with $\mathcal{O}_Y(-X) \cong \mathcal{O}_Y(-2)$ we get

$$H^0\mathcal{N}_{\Pi|Y} \otimes \mathcal{O}_Y(-X) = H^0\mathcal{N}_{\Pi|Y} \otimes \mathcal{O}_Y(-X) = 0 \qquad \text{and} \qquad H^1\mathcal{N}_{\Pi|Y} \otimes \mathcal{O}_Y(-X).$$

Since $H^3(\Omega^1_Y(-X)) \cong H^1(\mathcal{T}_Y) = 36$, while $H^3(\Omega^1_{\tilde{Y}}(-\tilde{X}_{16})) \cong H^1(\mathcal{T}_{\tilde{Y}}) = 33$ the Leray spectral sequence yields

$$H^i\Omega^1_{\tilde{Y}}(-\tilde{X}_{16}) = 0, \text{ for } i = 0, 1, 2.$$

From the exact sequence

$$0 \longrightarrow \Omega^1_{\tilde{Y}}(-\tilde{X}_{16}) \longrightarrow \Omega^1_{\tilde{Y}} \longrightarrow \Omega^1_{\tilde{Y}} \otimes \mathcal{O}_{\tilde{X}_{16}} \longrightarrow 0$$

we conclude

$$H^1(\Omega^1_{\tilde{Y}} \otimes \mathcal{O}_{\tilde{X}_{16}}) \cong H^1\Omega^1_{\tilde{Y}} \cong \mathbb{C}^2.$$

Finally, the exact sequence

$$0 \longrightarrow \mathcal{N}^{\vee}_{\tilde{X}_{16}|\tilde{Y}} \longrightarrow \Omega^1_{\tilde{Y}} \otimes \mathcal{O}_{\tilde{X}_{16}} \longrightarrow \Omega^1_{\tilde{X}_{16}} \longrightarrow 0$$

yields

$$H^1\Omega^1_{\tilde{X}_{16}} \cong H^1(\Omega^1_{\tilde{Y}} \otimes \mathcal{O}_{\tilde{X}_{16}}) \cong \mathbb{C}^2.$$

The standard computation with help of [14, Example 3.2.12] yields that the Euler number $e(\tilde{X}_{16}) = -108$ (see also [24, Prop. 1.14]), so we can compute $h^{1,2}(\tilde{X}_{16})$. □

As another consequence of [A1] we obtain the following simple observation.

*Remark* 1.8. The web $W$ contains no rank-4 quadrics.

*Proof.* Suppose that $Q_0 \in W$ is a rank-4 quadric. Then it is a cone through the 3-space $\operatorname{sing}(Q_0)$ over a smooth quadric in $\mathbb{P}_3$. The latter contains no planes, so the 3-space $\operatorname{sing}(Q_0)$ and the plane $\Pi$ meet. On the other hand, since each point in $\operatorname{sing}(Q_0) \cap Q_1 \cap Q_2 \cap Q_3$ is a singularity of $X_{16}$, the assumption [A1] implies that $\operatorname{sing}(Q_0)$ meets $\Pi$ in exactly one point $P \in \operatorname{sing}(X_{16})$. Moreover, we have $\operatorname{sing}(Q_0) \cap Q_1 \cap Q_2 \cap Q_3 = \{P\}$.
Lemma 1.4.b yields that the quadrics $Q_1$, $Q_2$, $Q_3$ are smooth in $P$. By Bézout the intersection multiplicity of $\operatorname{sing}(Q_0)$, $Q_1$, $Q_2$, $Q_3$ in the point $P$ is 8. The latter exceeds the product of multiplicities of the varieties in question in the point $P$. From [11, Thm 6.3] we obtain the inequality:

$$\dim(\operatorname{sing}(Q_0) \cap \mathrm{T}_P Q_1 \cap \mathrm{T}_P Q_2 \cap \mathrm{T}_P Q_3) \geq 1\,. \tag{8}$$

To complete the proof, suppose that $\operatorname{sing}(Q_0)$ is the zero set of the coordinates $x_0$, $x_1$, $x_6$, $x_7$. Recall that $\Pi$ is given by vanishing of $x_0, \ldots, x_4$, so we have $P = (0 : \ldots : 1 : 0 : 0)$ and only 12 entries in the matrix $\mathfrak{q}_0$ do not vanish.

The point $P$ is a node on $X_{16}$, so $\dim(\mathrm{T}_P Q_1 \cap \mathrm{T}_P Q_2 \cap \mathrm{T}_P Q_3) = 4$. Consider the affine chart $x_5 = 1$. The inequality (8) implies that there exists a nonzero $v := (0, 0, v_2, v_3, v_4, 0, 0)$ in the 4-dimensional intersection of the tangent spaces. Furthermore, all quadrics in question contain $\Pi$, so the 4-space contains the vectors $(0, \ldots, 0, 1, 0)$ and $(0, \ldots, 0, 1)$. Consequently, a parametrization of $\mathrm{T}_P Q_1 \cap \mathrm{T}_P Q_2 \cap \mathrm{T}_P Q_3$ is given by the map

$$(\lambda_1, \ldots, \lambda_4) \mapsto \lambda_1 v + \lambda_2 w + \lambda_3(0, \ldots, 1, 0) + \lambda_3(0, \ldots, 1),$$

where $w := (w_0, \ldots, w_4, 0, 0)$.

Finally, direct computation shows that intersection of the tangent cones $\mathrm{C}_P Q_0$, $\mathrm{T}_P Q_1$, $\mathrm{T}_P Q_2$, $\mathrm{T}_P Q_3$ consists of two planes. The latter is impossible because we assumed the point $P$ to be a node of $X_{16}$. Contradiction. □

## 2. Projection from the plane

Here we maintain the notation of the previous section. Moreover, we assume that [A1] holds and

**[A2]:** *no 4 singular points of $X_{16}$ lie on a line.*

In view of Lemma 1.4.a it seems natural to ask whether the assumption [A1] implies [A2]. The example below shows that this is not the case.

*Example* 2.1. Consider the following $8 \times 8$ symmetric matrices

$$\mathfrak{q}_0 := \begin{bmatrix} 0 & -4 & 4 & 0 & 1 & -2 & 0 & 1 \\ -4 & 4 & 4 & 3 & -3 & 2 & 2 & -2 \\ 4 & 4 & 4 & 1 & -1 & 0 & -1 & 0 \\ 0 & 3 & 1 & -2 & -1 & -2 & -1 & 2 \\ 1 & -3 & -1 & -1 & 2 & 0 & 0 & 0 \\ -2 & 2 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 2 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \mathfrak{q}_1 := \begin{bmatrix} -2 & 2 & -1 & -3 & 0 & 0 & 0 & -2 \\ 2 & 0 & -4 & 1 & 1 & 4 & -3 & 2 \\ -1 & -4 & 2 & 3 & 1 & 1 & 0 & -1 \\ -3 & 1 & 3 & -2 & -3 & 1 & -3 & 1 \\ 0 & 1 & 1 & -3 & 2 & -2 & 0 & 0 \\ 0 & 4 & 1 & 1 & -2 & 0 & 0 & 0 \\ 0 & -3 & 0 & -3 & 0 & 0 & 0 & 0 \\ -2 & 2 & -1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathfrak{q}_2 := \begin{bmatrix} -4 & 1 & 1 & 1 & -2 & -1 & -1 & -1 \\ 1 & 4 & -1 & -1 & -3 & -3 & 0 & 1 \\ 1 & -1 & 2 & -4 & 0 & 2 & 2 & 1 \\ 1 & -1 & -4 & 2 & -1 & -1 & 1 & 1 \\ -2 & -3 & 0 & -1 & -4 & -2 & 0 & 0 \\ -1 & -3 & 2 & -1 & -2 & 0 & 0 & 0 \\ -1 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \mathfrak{q}_3 := \begin{bmatrix} -4 & -1 & -4 & 3 & -1 & 4 & 1 & 0 \\ -1 & 4 & -4 & -3 & 0 & 3 & -1 & 0 \\ -4 & -4 & 0 & 1 & 0 & 1 & 1 & 1 \\ 3 & -3 & 1 & 2 & 2 & 1 & 0 & -2 \\ -1 & 0 & 0 & 2 & 4 & 3 & 0 & 0 \\ 4 & 3 & 1 & 1 & 3 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

By direct computation with help of [15], the intersection in $\mathbb{P}_7$ of the quadrics defined by the above matrices has 10 isolated singularities on the plane $\Pi$ and is smooth elsewhere. In the same way one checks that 4 singular points of the intersection in question lie on the line $(0 : \ldots : 0 : x_6 : x_7)$ and are given by the equation

$$19x_6^4 + 102x_6^3x_7 + 189x_6^2x_7^2 + 137x_6x_7^3 + 27x_7^4 = 0.$$

In this section we study the projection $X_{16} \setminus \Pi \ni (x_0 : \ldots : x_7) \mapsto (x_0 : \ldots : x_4) \in \mathbb{P}_4$ from the plane $\Pi$. Observe that the map in question lifts to a regular map

$$\pi : \tilde{X}_{16} \longrightarrow \mathbb{P}_4 \tag{9}$$

given by the linear system $|H - S|$, where $H$ is the pullback of a hyperplane section under the blow-up $\sigma : \tilde{X}_{16} \to X_{16}$, and $S$ stands for the strict transform of $\Pi$.

**Lemma 2.2.** *We have the following intersection numbers:*

$$\begin{aligned} &H^3 = 16, \\ &H^2 \cdot S = 1, \\ &H \cdot S^2 = -3, \\ &S^3 = -1, \\ &(H - S)^3 = 5. \end{aligned}$$

*Proof.* The first two statements are obvious. The intersection number $H \cdot S^2$ equals the intersection number in $S$ of the restrictions $H|_S$, $S|_S$. Since $S$ is a blow–up of the plane $\Pi$ in 10 points, the restriction $H|_S$ is the pullback $l$ of a line in $\Pi$. Moreover, $S|_S$ is the normal bundle of $S$ in the

Calabi–Yau manifold $\tilde{X}_{16}$. Hence it is the canonical divisor $K_S = -3l + \sum_1^{10} E_i$, where $E_1, \ldots, E_{10}$ are the 10 exceptional curves (see (7)). Finally, we have

$$H \cdot S^2 = (l \cdot (-3l + \sum_1^{10} E_i))_S = -3.$$

Similarly, $S^3 = ((-3l + \sum_1^{10} E_i)^2)_S = 9 - 10 = -1$. The last statement follows from Newton's formula. □

To simplify our notation we put $\underline{x} := (x_0 : \ldots : x_4) \in \mathbb{P}^4$ and define the following matrices :

$$\mathfrak{a}(\underline{x}) := \begin{bmatrix} \mathfrak{l}_0\underline{x} & \mathfrak{l}_1\underline{x} & \mathfrak{l}_2\underline{x} & \mathfrak{l}_3\underline{x} \\ \mathfrak{m}_0\underline{x} & \mathfrak{m}_1\underline{x} & \mathfrak{m}_2\underline{x} & \mathfrak{m}_3\underline{x} \\ \mathfrak{n}_0\underline{x} & \mathfrak{n}_1\underline{x} & \mathfrak{n}_2\underline{x} & \mathfrak{n}_3\underline{x} \end{bmatrix}, \quad \mathfrak{A}(\underline{x}) := \begin{bmatrix} \underline{x}^T \underline{\mathfrak{q}_0} \underline{x} & \\ \underline{x}^T \underline{\mathfrak{q}_1} \underline{x} & \mathfrak{a}(\underline{x})^T \\ \underline{x}^T \underline{\mathfrak{q}_2} \underline{x} & \\ \underline{x}^T \underline{\mathfrak{q}_3} \underline{x} & \end{bmatrix}. \tag{10}$$

Observe that the following equality holds (cf. [2, p. 30])

$$\mathfrak{a}(\underline{x})y = \mathfrak{b}(y)\underline{x}. \tag{11}$$

Let $\underline{Q_i}$ be the quadratic form associated to the matrix $\underline{\mathfrak{q}_i}$ and let $\mathcal{C}_i$ denote the cubic given by the degree-3 minor of the matrix $\mathfrak{a}(\underline{x})$ obtained by deleting its $i$-th column, e.g.

$$\mathcal{C}_0 := \det \begin{bmatrix} \mathfrak{l}_1\underline{x} & \mathfrak{l}_2\underline{x} & \mathfrak{l}_3\underline{x} \\ \mathfrak{m}_1\underline{x} & \mathfrak{m}_2\underline{x} & \mathfrak{m}_3\underline{x} \\ \mathfrak{n}_1\underline{x} & \mathfrak{n}_2\underline{x} & \mathfrak{n}_3\underline{x} \end{bmatrix}.$$

**Lemma 2.3.** *a) The image of $\tilde{X}_{16}$ under $\pi$ is the quintic $X_5$ given by the equation*

$$\det(\mathfrak{A}(\underline{x})) = \mathcal{C}_0 \cdot \underline{Q_0} - \mathcal{C}_1 \cdot \underline{Q_1} + \mathcal{C}_2 \cdot \underline{Q_2} - \mathcal{C}_3 \cdot \underline{Q_3} = 0\,. \tag{12}$$

*b) The image of $S$ under $\pi$ is the (smooth) Bordiga sextic $\mathcal{B} \subset \mathbb{P}_4$ given by vanishing of the cubics $\mathcal{C}_0, \ldots, \mathcal{C}_4$ (i.e. all $3 \times 3$ minors of the matrix $\mathfrak{a}(\underline{x})$). Moreover, the map $\pi|_S : S \to \mathcal{B}$ is an isomorphism.*

*Proof.* Obviously, the restriction of the quadric $\sum_0^3 \alpha_i Q_i$ to the 3-space

$$\mathrm{span}\{\underline{x}, \Pi\} = \{(\mu_0 x_0 : \ldots : \mu_0 x_3 : \mu_0 x_4 : \mu_1 : \mu_2 : \mu_3) \mid (\mu_0 : \mu_1 : \mu_2 : \mu_3) \in \mathbb{P}_3\}$$

is given by the polynomial

$$(\sum_0^3 \alpha_i \underline{x}^T \underline{\mathfrak{q}_i} \underline{x})\mu_0^2 + 2(\sum_0^3 \alpha_i(\mathfrak{l}_i\underline{x}))\mu_0\mu_1 + 2(\sum_0^3 \alpha_i(\mathfrak{m}_i\underline{x}))\mu_0\mu_2 + 2(\sum_0^3 \alpha_i(\mathfrak{n}_i\underline{x}))\mu_0\mu_3. \tag{13}$$

a) Observe that $\underline{x} \in \mathbb{P}_4 \setminus \pi(S)$ lies in the image of $X_{16}$ under the projection from $\Pi$ iff the planes residual to $\Pi$ in the intersections of the quadrics $Q_i$ with the 3-space $\mathrm{span}\{\underline{x}, \Pi\}$ intersect. By (13), the latter is equivalent to the vanishing $\det(\mathfrak{A}(\underline{x})) = 0$. Laplace formula completes the proof.
b) From (13) we obtain that the condition

$$\sum_0^3 \alpha_i(\mathfrak{l}_i\underline{x}) = \sum_0^3 \alpha_i(\mathfrak{m}_i\underline{x}) = \sum_0^3 \alpha_i(\mathfrak{n}_i\underline{x}) = 0$$

is satisfied iff the restriction $(\sum_0^3 \alpha_i Q_i)|_{\mathrm{span}\{\underline{x}, \Pi\}}$ is the double plane $2\Pi$. The latter holds precisely when $\underline{x}$ lies in the image of $\Pi$ under the projection in question.

It is well known that, for a generic $4 \times 3$ matrix whose entries are linear forms in five variables, the surface given by the vanishing of $3 \times 3$ minors is $\mathbb{P}_2$ blown-up in 10 points (see e.g. [2]). Still, it is not always the case (see e.g. [30]). To see that our surface is indeed the (smooth)

Bordiga sextic, observe that the linear system $|H - S|$ restricts on $S$ to the complete linear system $|4l - \sum_{i=1}^{i=10} E_i|$. We apply [4, Lemma 2.9.1] to show that the system in question embeds $S$ into $\mathbb{P}_4$ as the (smooth) Bordiga sextic. By Lemma 1.4.a no cubic contains all singularities of $X_{16}$. Suppose that 8 singularities of $X_{16}$ lie on a conic. Then its product with the line through the remaining two singular points is a cubic containing $\mathrm{sing}(X_{16})$. Consequently the existence of such a conic is ruled out by Lemma 1.4.a. Finally no 4 singularities lie on a line by the assumption [A2]. $\square$

*Remark* 2.4. a) Observe that, since the (scheme–theoretic) intersection $\mathcal{B}$ of the zeroes of the degree-3 minors of the matrix $\mathfrak{a}(\underline{x})$ is smooth, we have

$$\mathrm{rank}(\mathfrak{a}(\underline{x})) = 2 \quad \text{ for every } \underline{x} \in \mathcal{B}.$$

b) The rational curves $E_1, \ldots, E_{10} \subset \tilde{X}_{16}$ are mapped by $\pi$ to lines in $\mathbb{P}_4$ contained in the Bordiga sextic. Indeed, we have $(H - S) \cdot E_j = ((4l - \sum E_i) \cdot E_j)_S = 1$ for $j = 1, \ldots, 10$.
Geometrically, points on such a line $\subset \mathcal{B}$ correspond to the 3-spaces in the 4-space $\mathrm{T}_P X_{16}$, where $P$ is a node of $X_{16}$, that contain the plane $\Pi$.

We introduce the following notation:

$$U := \tilde{X}_{16} \setminus (S \cup \bigcup_{V \text{ linear}, V \subset X_{16}, V \cap \Pi \neq \emptyset} \sigma^{-1}(V)).$$

**Lemma 2.5.** *Suppose that [A1], [A2] hold.*
*a) The map $\pi|_U$ is an isomorphism onto the image and we have the equality $\pi(U) = (X_5 \setminus \mathcal{B})$.*
*b) The inclusion $\mathrm{sing}(X_5) \subsetneq \mathcal{B}$ holds. In particular, the quintic $X_5$ is normal.*

*Proof.* a) Fix $P \in U$. Then $\sigma(P) \notin \Pi$. Since $X_{16}$ is an intersection of quadrics we have the equality

$$\mathrm{span}(\sigma(P), \Pi) \cap X_{16} = \Pi \cup \{\sigma(P)\}, \text{ where } \sigma(P) \notin \Pi$$

which implies that $\pi|_U$ is injective and the linear map $d_P\pi$ is an isomorphism.

We claim that

$$\pi(\tilde{X}_{16} \setminus U) = \mathcal{B}.$$

Let $V \subset X_{16}$, $V \not\subseteq \Pi$ be a linear subspace such that $V \cap \Pi \neq \emptyset$. Let $\sigma(P_1) \in (V \setminus \Pi)$ and let $\sigma(P_2) \in (V \cap \Pi)$. By definition of $\pi$ all points from $\mathrm{span}(\sigma(P_1), \sigma(P_2)) \setminus \{\sigma(P_2)\}$ lie in one fiber of $\pi$. On the other hand, the proper transform of the line $\mathrm{span}(\sigma(P_1), \sigma(P_2))$ under $\sigma$ meets $S$. Since $\pi$ maps that proper transform of the line in question to one point and $\pi(P_2) \in \mathcal{B}$ we have $\pi(P_1) \in \mathcal{B}$, and we obtain the claim.

It remains to show the inclusion

$$\pi(U) \subset (X_5 \setminus \mathcal{B}).$$

Suppose that $\pi(P_3) = \pi(P_4)$, where $P_3 \in \tilde{X}_{16} \setminus U$ and $P_4 \in U$. If $\sigma(P_3) \in \mathrm{reg}(X_{16})$, then the line $\mathrm{span}(\sigma(P_3), \sigma(P_4))$ is tangent to $X_{16}$ in $\sigma(P_3)$ and meets it in $\sigma(P_4)$. In particular, it is contained in each quadric of the system $W$, so $\mathrm{span}(\sigma(P_3), \sigma(P_4)) \subset X_{16}$ and $P_4 \notin U$. Contradiction.
Similar argument yields contradiction when $\sigma(P_3) \in \mathrm{sing}(X_{16})$.
b) By [A1] and part a) we know that $\mathrm{sing}(X_5) \subset \mathcal{B}$. Suppose that $\mathrm{sing}(X_5) = \mathcal{B}$. Since $\mathcal{B}$ is smooth, Lemma 2.3.a implies that $\det(\mathfrak{A}(\underline{x})) \in \mathrm{I}(\mathcal{B})^2$. The latter is impossible because the ideal $\mathrm{I}(\mathcal{B})$ is generated by the cubics $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$.
Finally $X_5$ is a 3-dimensional hypersurface with at most 1-dimensional singularities, so it is normal. $\square$

After those preparations we can study higher-dimensional fibers of $\pi$.

**Lemma 2.6.** *a) The map $\pi$ has no two-dimensional fibers and its only one-dimensional fibers are proper transforms of lines on $X_{16}$ that meet $\Pi$ but are not contained in $\Pi$.*
*b) The following equality holds*

$$\operatorname{sing}(X_5) := \{\underline{x} \in \mathcal{B} \,:\, \operatorname{rank}(\mathfrak{A}(\underline{x})) \leq 2\}\,. \tag{14}$$

*c) The map $\pi$ has only finitely many one–dimensional fibers.*

*Proof.* a) As we have already shown in the proof of Lemma 2.5 the proper transform of each line on $X_{16}$ that meets $\Pi$ but is not contained in $\Pi$ lies in a fiber of $\pi$.

The regular map $\pi$ is birational and its image is normal, so we can apply Zariski's Main Theorem [17, Thm 5.2] to see that the map $\pi$ has connected fibers. Moreover, by Lemma 2.3.b

$$\text{each fiber of } \pi \text{ meets the surface } S \text{ in at most one point.} \tag{15}$$

Let $F$ be a fiber of $\pi$ such that $\dim(F) \geq 1$. Let $P_1, P_2 \in (F \setminus S)$. Then the 3-spaces $\operatorname{span}(\sigma(P_1), \Pi)$, $\operatorname{span}(\sigma(P_2), \Pi)$ coincide, so the line $\operatorname{span}(\sigma(P_1), \sigma(P_2))$ meets the plane $\Pi$. Obviously, the intersection point does not coincide with $P_1$, $P_2$. Since $X_{16}$ is intersection of quadrics, we have $\operatorname{span}(\sigma(P_1), \sigma(P_2)) \subset X_{16}$, which implies that

$$\operatorname{span}(\sigma(P_1), \sigma(P_2)) \subset \sigma(F)\,.$$

Suppose that the fiber $F$ contains a point $P_3 \notin S$ such that $\sigma(P_3) \notin \operatorname{span}(\sigma(P_1), \sigma(P_2))$. Then, arguing as in (2), we show that $\operatorname{span}(\sigma(P_1), \sigma(P_3))$ is a line contained in $\sigma(F)$ and meeting the plane $\Pi$. But, (15) implies that the proper transforms (under the blow-up $\sigma$) of two lines meeting $\Pi$ in different points cannot lie in the same fiber of $\pi$. Consequently, by (15), the image $\sigma(F)$ is a plane in $X_{16}$ that intersects $\Pi$ in precisely one point. Observe that the planes $\sigma(F)$, $\Pi$ meet in a singularity of $X_{16}$. Let $H$ be the pullback of a hyperplane section under the blow-up $\sigma$ and let $\widetilde{\sigma(F)}$ denote the proper transform of $\sigma(F)$. If we put $\tilde{l}$ (resp. $\tilde{m}$) to denote the proper transform of a line in $\sigma(F)$ (resp. in $\Pi$) that runs through no singularities of $X_{16}$, then we obtain the following table of intersection numbers.

| | $\widetilde{\sigma(F)}$ | $S$ | $H$ |
|---|---|---|---|
| $\tilde{l}$ | $-3$ | $0$ | $1$ |
| $\tilde{m}$ | $0$ | $-3$ | $1$ |
| $H^2$ | $1$ | $1$ | $16$ |

The resulting matrix has non-zero determinant, so Picard number of $\tilde{X}_{16}$ is at least 3, which is impossible by Lemma 1.7. This contradiction shows that the fiber $F$ coincides with the proper transform of the line $\operatorname{span}(\sigma(P_1), \sigma(P_2))$.
b) As in the proof of Lemma 2.3, we see that the line through the points $(\underline{x}, x_5, x_6, x_7)$ and $(0, x_5', x_6', x_7')$ is contained in $X_{16}$ iff for any $\lambda \in \mathbb{C}$ and $i = 0, \ldots, 3$ we have

$$\underline{x}^T \underline{\mathfrak{q}}_i \underline{x} + 2(\mathfrak{l}_i\underline{x}, \mathfrak{m}_i\underline{x}, \mathfrak{n}_i\underline{x})(x_5, x_6, x_7)^T + 2\lambda(\mathfrak{l}_i\underline{x}, \mathfrak{m}_i\underline{x}, \mathfrak{n}_i\underline{x})(x_5', x_6', x_7')^T = 0.$$

Fix $\underline{x} \in \mathcal{B}$. From Remark 2.4.a we know that $\operatorname{rank}(\mathfrak{a}(\underline{x})) = 2$. Consequently, there exist points $(x_5, x_6, x_7)$ and $(x_5', x_6', x_7')$ such that the line spanned by $(\underline{x}, x_5, x_6, x_7)$ and $(0, x_5', x_6', x_7')$ is contained in $X_{16}$ if and only if $\operatorname{rank}(\mathfrak{A}(\underline{x})) = 2$.
c) Assume to the contrary that the map $\pi$ contracts infinitely many lines. Then there is a ruled surface $G \subset \tilde{X}_{16}$ such that the fibers of $G$ are contracted by $\pi$. Let $l$ (resp. $E_i$) be the class of a (general) fiber of $G$, (resp. of an exceptional curve of the blow-up $\sigma$). We have the following

intersection numbers

(16)

| | $S$ | $G$ | $H$ |
|---|---|---|---|
| $l$ | $1$ | $-2$ | $1$ |
| $E_i$ | $-1$ | $\nu$ | $0$ |

The above table yields immediately that $H$ and $S$ are linearly independent in $\mathrm{Pic}(\tilde{X}_{16}) \otimes \mathbb{Q}$. Since $h^{1,1}(\tilde{X}_{16}) = 2$, we can find $d_H, d_S \in \mathbb{Q}$ such that $G \sim_{\mathrm{num}} d_H H + d_S S$. From (16) we obtain

$$G \sim_{\mathrm{num}} (\nu - 2)H - \nu S.$$

Therefore Lemma 2.2 yields the equality

$$(H - S)^2 \cdot G = 5\nu - 22.$$

As the divisor $G$ is contracted by $\pi$ we conclude that $\nu = \frac{22}{5}$, which is impossible by (16). $\square$

In particular, Lemma 2.6 implies that the map $\pi : \tilde{X}_{16} \longrightarrow X_5$ is a resolution of singularities of the quintic $X_5$. As $\pi$ contracts only finitely many curves (i.e. the singular locus of $X_5$ is zero-dimensional), it is in fact a small resolution that introduces exactly one copy of $\mathbb{P}_1$ over each singularity.

The lemma below gives a simple criterion when the quintic $X_5$ is nodal.

**Lemma 2.7.** *All singularities of the quintic $X_5$ are nodes iff the set $\mathrm{sing}(X_5)$ consists of $46$ points.*

*Proof.* Let $\mu(\cdot)$ stand for the Milnor number. Lemma 2.5 yields that the regular map $\pi : \tilde{X}_{16} \longrightarrow X_5$ is birational. By Lemma 2.6 it contracts only the lines in $X_{16}$ that intersect the plane $\Pi$. The contracted lines are pairewise disjoint, so we obtain

$$-108 - \#(\mathrm{sing}(X_5)) = e(X_5) = -200 + \sum_{P \in \mathrm{sing}(X_5)} \mu(P, X_5),$$

where the second equality results from [10, Cor. 5.4.4]. To complete the proof recall that the Milnor number of a singularity is 1 iff the singularity in question is an $\mathrm{A}_1$ point. $\square$

## 3. Restriction of the Bordiga conic bundle

In this section we maintain the assumptions and notation of the previous one, i.e. we assume that [A1], [A2] hold. In particular, *the scheme-theoretic intersection of the zeroes of the degree-3 minors of the matrix $\mathfrak{a}(\underline{x})$ is smooth* (see (10)) and *the locus $\{y \in \mathbb{P}_4 : \mathrm{rank}(\mathfrak{b}(y)) = 2\}$ consists of 10 points*. Moreover, we make the following **assumption**:

**[A3]:** *the set $\{\underline{x} \in \mathcal{B} : \mathrm{rank}(\mathfrak{A}(\underline{x})) \leq 2\}$ consists of 46 points* .

One can show (see [2, Ex. 3 on p. 35]) that the rational map

$$\mathbb{P}_4 \setminus \mathcal{B} \ni \underline{x} \mapsto (\mathcal{C}_0(\underline{x}) : -\mathcal{C}_1(\underline{x}) : \mathcal{C}_2(\underline{x}) : -\mathcal{C}_3(\underline{x})) \in \mathbb{P}_3 \tag{17}$$

lifts to a regular map (so-called Bordiga conic bundle - see [2, Ex. 3 on p. 35])

$$\Phi : \mathrm{Bl}_{\mathcal{B}}\mathbb{P}_4 \to \mathbb{P}_3.$$

that is generically a conic-bundle ([ibid., Prop. 2.1]). The map $\Phi$ is the projection onto the second factor from the closure of the graph of the rational map defined by (17) (see also (11)) i.e. from the set

$$\{(\underline{x}, y) \in \mathbb{P}_4 \times \mathbb{P}_3 : \mathfrak{b}(y)\underline{x} = 0\}. \tag{18}$$

By Lemma 1.4.d it has exactly ten 2-dimensional fibers over the points $y \in \mathbb{P}_3$ such that $\mathrm{rank}(\mathfrak{b}(y)) = 2$. Such a fiber is the plane

$$\Phi^{-1}(y) = \{(\underline{x}, y) : \mathfrak{b}(y)\underline{x} = 0\}. \tag{19}$$

Observe that restrictions of the cubics polynomials $\mathcal{C}_i$ to the plane $\{\mathfrak{b}(y)\underline{x} = 0\}$ are proportional, so the plane cuts $\mathcal{B}$ along a cubic curve (see also [2, Ex. 3 on p. 35]).
The remaining fibers $\Phi^{-1}(y)$ are 3-secant lines to $\mathcal{B}$. They are given by (19) with $\mathrm{rank}(\mathfrak{b}(y)) = 3$.

In Sect. 1 we studied the map $\tilde{X}_{16} \longrightarrow X_5$. By Lemma 2.7 the quintic $X_5$ admits another small resolution of singularities

$$\psi : \tilde{X}_5 \longrightarrow X_5 \tag{20}$$

obtained by blowing–up the Bordiga surface $\mathcal{B}$. The strict transform $S_1$ of $\mathcal{B}$ is a plane blown–up in 56 points (some of the 46 points that are centers of the second blow-up may lie on the exceptional curves of the first blow–up). We put $F_1, \ldots, F_{46}$ to denote the exceptional curves of the small resolution in question. Then, the two resolutions differ by flops of the 46 smooth rational curves $L_1, \ldots, L_{46} \subset \tilde{X}_{16}$ and $F_1, \ldots, F_{46} \subset \tilde{X}_5$.

The restriction of the conic bundle $\Phi$ induces the regular map

$$\phi : \tilde{X}_5 \longrightarrow \mathbb{P}_3.$$

This regular map is given by the linear system $|3H_1 - S_1|$ on $\tilde{X}_5$, where $H_1$ is pullback of the hyperplane section $\mathcal{O}_{\mathbb{P}_4}(1)$. We have the following intersection numbers

**Lemma 3.1.**

$$\begin{aligned} H_1^3 &= 5, \\ H_1^2 \cdot S_1 &= 6, \\ H_1 \cdot S_1^2 &= -2, \\ S_1^3 &= -47, \\ (3H_1 - S_1)^3 &= 2. \end{aligned}$$

*Proof.* The first two statements follow from the fact that $\deg(X_5) = 5$ and $\deg(\mathcal{B}) = 6$. The others can be obtained from the equalities

$$H_1|_{S_1} = 4l - \sum_1^{10} \psi^*(\pi(E_i)), \quad S_1|_{S_1} = -3l + \sum_1^{10} \psi^*(\pi(E_i)) + \sum_1^{46} F_j. \tag{21}$$

where $l$ is the pull-back of $\mathcal{O}_{\mathbb{P}_1}(1)$ under both blow-ups. Recall (Remark 2.4.b) that the curves $\pi(E_1)$,..., $\pi(E_{10})$ are lines on $\mathcal{B}$. $\square$

Since $\phi$ is surjective, as an immediate consequence of Lemma 3.1 we obtain

**Corollary 3.2.** *The mapping $\phi$ is generically 2:1.*

In order to obtain a precise description of fibers of $\phi$ we will need the following lemma (compare [24]):

**Lemma 3.3.** *A point $z \in \tilde{X}_5$ is mapped by $\phi$ to $y \in \mathbb{P}_3$ iff the 3-space* $\mathrm{span}((\psi(z) : 0 : 0 : 0), \Pi)$ *is contained in the quadric $Q(y) := \sum_i y_i Q_i$.*

*Proof.* Observe that for any $x = (\underline{x} : x_5 : x_6 : x_7) \in \mathrm{span}((\underline{x} : 0 : 0 : 0), \Pi)$ we have

$$x^T \mathfrak{q}(y) x = \underline{x}^T \underline{\mathfrak{q}}(y) \underline{x} + 2(x_5, x_6, x_7)\mathfrak{b}(y)\underline{x} \tag{22}$$

($\Leftarrow$): Put $\underline{x} = \psi(z)$ in (22) to obtain

$$\psi(z)^T \underline{\mathfrak{q}}(y) \psi(z) = -2(x_5, x_6, x_7)\mathfrak{b}(y)\psi(z) \quad \text{for all } x_5, x_6, x_7 \in \mathbb{C}.$$

The latter implies $\mathfrak{b}(y)\psi(z) = 0$ and (see (19)) the equality $\phi(z) = y$.
($\Rightarrow$): Suppose that $z \in \tilde{X}_5 \setminus S_1$. From $\phi(z) = y$ we get $\mathfrak{b}(y)\psi(z) = 0$. By (22) we have

$$x^T \mathfrak{q}(y) x = \psi(z)^T \underline{\mathfrak{q}}(y)\psi(z) \qquad \text{for all } x = (\psi(z) : x_5 : x_6 : x_7) \in \operatorname{span}(\psi(z), \Pi).$$

But (see (17)), we can assume that $y = (\mathcal{C}_0(\psi(z)) : \ldots : -\mathcal{C}_3(\psi(z)))$. Therefore, Lemma 2.3.a yields the equalities $\psi(z)^T \underline{\mathfrak{q}}(y)\psi(z) = \det(\mathfrak{A}(\psi(z))) = 0$. In this way we have shown the inclusion

$$\{(\underline{x}, y) \in \tilde{X}_5 : \mathfrak{b}(y)\underline{x} = 0\} \subset \{(\underline{x}, y) \in \mathbb{P}_4 \times \mathbb{P}_3 : \operatorname{span}((\underline{x} : 0 : 0 : 0), \Pi) \subset Q(y)\},$$

which completes the proof. $\square$

Recall, that we have the map $(\psi \circ (\pi|_S)^{-1} \circ \sigma) : S_1 \to \mathcal{B} \backsimeq S \to \Pi$. In the lemma below we put $\hat{l}$ (resp. $\hat{E}_1, \ldots, \hat{E}_{10}$) to denote the pullback of $\mathcal{O}_\Pi(1)$ (resp. of the exceptional divisors (7)) to $S_1$.

**Lemma 3.4.** *An irreducible curve $D \subset S_1$ is contracted by $\phi$ iff (up to a relabelling of the divisors $\hat{E}_1, \ldots, \hat{E}_{10}$ and $F_1, \ldots, F_{46}$) it belongs to one of the following linear systems*

a) $|\hat{E}_1 - F_1 - F_2 - F_3 - F_4|$,
b) $|\hat{l} - \hat{E}_1 - \hat{E}_2 - \hat{E}_3 - F_1 - F_2 - F_3|$,
c) $|2\hat{l} - \hat{E}_1 - \ldots - \hat{E}_7 - F_1 - F_2|$,
d) $|3\hat{l} - 2\hat{E}_1 - \hat{E}_2 - \ldots - \hat{E}_9 - F_1 - \ldots - F_5|$.

*In the cases (a)–(c) the curve in question is the proper transform of a line in $\mathcal{B}$, whereas the case (d) corresponds to a conic in the intersection of $\mathcal{B}$ with the plane $\{\mathfrak{b}(y)\underline{x} = 0\}$, where $\operatorname{rank}(\mathfrak{b}(y)) = 2$. In particular, if the intersection $\mathcal{B} \cap \{\mathfrak{b}(y)\underline{x} = 0\}$ is an irreducible cubic, then its proper transform is not contracted by $\phi$.*

*Proof.* Recall that $\phi = \Phi|_{\tilde{X}_5}$ and the fibers of $\Phi$ are lines and planes given by (19).

Before we prove the claim, we study two-dimensional fibers of $\Phi$. Let $\operatorname{sing}(X_{16}) = \{P_1, \ldots, P_{10}\}$. By (3) for each singularity $P_i$ there exists a unique point $y^{(i)} \in \mathbb{P}_3$ such that $\mathfrak{c}(P_i)y^{(i)} = 0$. Then, by (2), we have $\operatorname{rank}(\mathfrak{b}(y^{(i)})) = 2$.
Lemma 1.4.a yields that for each $i \in \{1, \ldots, 10\}$ there is a unique degree-three curve $C_i \subset \Pi$ such that $P_j \in C_i$, for $j \neq i$. Let $\tilde{C}_i := \sigma^* C_i - \sum_{j \neq i} E_j \in |3l - \sum_{j \neq i} E_j|$ be the corresponding curve on $S$. By direct computation the following equality holds

$$\pi(\tilde{C}_i) = \mathcal{B} \cap \{\underline{x} \in \mathbb{P}_4 : \mathfrak{b}(y^{(i)})\underline{x} = 0\} \tag{23}$$

In general, cubics $C_i$ are smooth, and the curves $\pi(\tilde{C}_i) \subset \mathcal{B}$ are also smooth planar cubics. We have the following possible degenerations:
(i) The curve $C_i$ is irreducible, but $\operatorname{sing}(C_i) = \{P_{j_0}\}$ for a $j_0 \neq i$. Then the exceptional curve $E_{j_0}$ is a component of the curve $\tilde{C}_i := \sigma^* C_i - \sum_{j \neq i} E_j$ and the curve $\tilde{C}_i - E_{j_0}$ is irreducible. By Remark 2.4.b the image $\pi(E_{j_0})$ is a line on $\mathcal{B}$, whereas $\pi(\tilde{C}_i)$ is a smooth conic. In this way we obtain a decomposition of $\mathcal{B} \cap \{\underline{x} \in \mathbb{P}_4 : \mathfrak{b}(y^{(i)})\underline{x} = 0\}$. Observe that for a given integer $i \neq j_0$ there exists at most one cubic in $|\mathcal{O}_\Pi(3) - \sum_{j \neq i} E_j - E_{j_0}|$.
(ii) The cubic $\tilde{C}_i$ is union of a line and a smooth conic. Then, by [A2] and Lemma 1.4.a the line contains two (resp. three) singularities of $X_{16}$ and the conic contains 7 (resp. 6) of them.
(iii) The curve $\tilde{C}_i$ can be union of three lines. The assumption [A2] yields that each line contains three singularities of $X_{16}$.
In this way (up to a permutation of the points in $P_1, \ldots, P_9$), we obtain the following possibilities

for the decomposition of the cubic (23) for $i = 10$:

$$
(24) \qquad \begin{aligned}
&(3l - 2E_1 - E_2 - \cdots - E_9) + E_1,\\
&(l - E_1 - E_2) + (2l - E_3 - \cdots - E_9),\\
&(l - E_1 - E_2 - E_3) + (2l - E_4 - \cdots - E_9),\\
&(l - E_1 - E_2 - E_3) + (2l - E_3 - \cdots - E_9) + E_3,\\
&(l - E_1 - E_2 - E_3) + (l - E_4 - E_5 - E_6) + (l - E_7 - E_8 - E_9).
\end{aligned}
$$

After those preparations we can prove the lemma. Assume that an irreducible curve $D \subset S_1$ is contained in $\phi^{-1}(y)$ for a point $y \in \mathbb{P}_3$ . The map $\phi|_{S_1} : S_1 \to \mathbb{P}_3$ is given by the linear system

$$
(25) \qquad |15\hat{l} - 4\sum_1^{10} \hat{E}_i - \sum_1^{46} F_j|,
$$

so $D \neq F_j$ for each $j \leq 46$.

Suppose that $\mathrm{rank}(\mathfrak{b}(y)) = 2$. We can assume that $D \subset \phi^{-1}(y^{(10)})$. Then $\psi(D) \subset \mathcal{B}$ is a component of (23). If $\psi(D)$ is image under $\pi$ of a curve from the system $|3l - 2E_1 - E_3 - \cdots - E_9|$, then we have

$$
\deg(\psi(D)) = (3l - 2E_1 - E_3 - \cdots - E_9) \cdot (4l - \sum_1^{10} E_i) = 12 - 2 - 8 = 2.
$$

Let $\mathrm{sing}(X_5) \cap \psi(D) = \{\psi(F_1), \ldots, \psi(F_p)\}$. Since $D$ coincides with the proper transform of $\psi(D)$ under the blow-up $\psi$, we have

$$
D \in |3\hat{l} - 2\hat{E}_1 - \hat{E}_2 - \cdots - \hat{E}_9) - F_1 - \cdots - F_p|.
$$

and, by (25), the degree of $\phi(D)$ is $(5 - p)$. Consequently, the curve $D$ is contracted by $\phi$ iff $p = 5$.

In the following table we collect data on each curve considered in (24). In particular, the integer in the last column is the number of singularities of $X_5$ that lie on $\psi(D)$ provided $D$ is contracted by the map $\phi$:

| $\lvert\pi^{-1}(\psi(D))\rvert$ | $\deg(\psi(D))$ | $\#(\mathrm{sing}(X_5) \cap \psi(D))$ |
|---:|:---:|:---:|
| $3l - 2E_1 - E_2 - \cdots - E_9$ | 2 | 5 |
| $2l - E_1 - \ldots - E_6$ | 2 | 6 |
| $2l - E_1 - \ldots - E_7$ | 1 | 2 |
| $l - E_1 - E_2$ | 2 | 7 |
| $l - E_1 - E_2 - E_3$ | 1 | 3 |
| $E_1$ | 1 | 4 |

Finally, observe that for a point $y^{(i)} \in \mathbb{P}_3$, where $i = 1, \ldots 10$, the intersection

$$
(26) \qquad X_5 \cap \{\underline{x} \in \mathbb{P}_4 : \mathfrak{b}(y^{(i)})\underline{x} = 0\}
$$

is a degree–5 planar curve, so it is union of the cubic considered above and a conic (possibly reducible) that does not lie on $\mathcal{B}$. The points $\psi(F_j)$ are singular points of $X_5$, so they are also singular points of the quintic curve (26), which yields some extra constrains on the possible arrangements. Since a line contained in (26) intersects the residual quartic in four points, the line of the type $(l - E_1 - E_2)$ is never contracted. Similar argument rules out the conic $(2l - E_1 - \ldots - E_6)$. In this way we arrive at the cases (a)–(d) of the lemma.

Assume that $\mathrm{rank}(\mathfrak{b}(y)) = 3$. Then $D$ is the strict transform of a line $l_y \subset \mathcal{B}$. In particular, there exist $d, m_i, n_j \in \mathbb{Z}$ such that $D \in |d\hat{l} - \sum_1^{10} m_i\hat{E}_i - \sum_1^{46} n_jF_j|$. Since the curve $D$ is smooth

and rational, we have $n_j = 0$ or 1. Moreover, by the genus formula

$$(d\hat{l} - \sum_1^{10} m_i \hat{E}_i - \sum_1^{46} n_j F_j) \cdot ((d-3)\hat{l} - \sum_1^{10} (m_i - 1)\hat{E}_i - \sum_1^{46} (n_j - 1) F_j) = d^2 - 3d - \sum_1^{10} (m_i^2 - m_i) = -2.$$

Furthermore, the equality $4d - \sum_1^{10} m_i = 1$ holds because $l_y$ is a line on $\mathcal{B}$ (see also Lemma 2.3.b). Finally, since $D$ is contracted by the map given by the linear system $|3H_1 - S_1|$ we have

$$(15\hat{l} - 4\sum_1^{10} \hat{E}_i - \sum_1^{46} F_j) \cdot (d\hat{l} - \sum_1^{10} m_i \hat{E}_i - \sum_1^{46} n_j F_j) = 15d - 4\sum_1^{10} m_i - \sum_1^{46} n_j = 0.$$

From the above we obtain the following equations

$$\begin{aligned} &\sum m_i^2 = d^2 + d + 1, \\ &\sum m_i = 4d - 1, \\ &4 - d = \sum n_j, \end{aligned}$$

where $n_j = 0, 1$. The solution $d = 3$, $m_1 = 2$, $m_i = 1$ for $i > 1$ is excluded by Lemma 1.4.a. The others correspond to the cases (a)–(c) of the lemma. $\square$

Now we are in position to prove

**Lemma 3.5.** *Let $y \in \mathbb{P}_3$ be a point such that $\operatorname{rank}(\mathfrak{b}(y)) = 3$. Then the fiber $\phi^{-1}(y)$ is 1-dimensional iff $\operatorname{rank}(\mathfrak{q}(y)) = 6$.*

*Proof.* By abuse of notation we put $\psi$ to denote the blow-up $\mathrm{Bl}_{\mathcal{B}}\mathbb{P}_4 \to \mathbb{P}_4$.

Assume that the line $\Phi^{-1}(y)$ is contracted by $\phi$. Then the set $\psi(\Phi^{-1}(y)) = \{\underline{x} \in \mathbb{P}_4 : \mathfrak{b}(y)\underline{x} = 0\}$ is a line on $X_5$. Observe that the linear space $\operatorname{span}(\{(\underline{x} : 0 : 0 : 0) : \underline{x} \in \psi(\Phi^{-1}(y))\}, \Pi)$ is 4-dimensional. By Lemma 3.3 the quadric $Q(y)$ contains the 4–space $\operatorname{span}(\{(\underline{x} : 0 : 0 : 0) : \underline{x} \in \psi(\Phi^{-1}(y))\}, \Pi)$, which yields $\operatorname{rank}(\mathfrak{q}(y)) \le 6$. Finally $\operatorname{rank}(\mathfrak{q}(y)) = 6$, because $\operatorname{rank}(\mathfrak{b}(y)) = 3$.

On the other hand, if $\operatorname{rank}(\mathfrak{q}(y)) = 6$, then $\operatorname{sing}(Q(y))$ is a line. Since $\operatorname{rank}(\mathfrak{b}(y)) = 3$, the line $\operatorname{sing}(Q(y))$ does not meet the plane $\Pi$. Put $L$ to denote the image of the line $\operatorname{sing}(Q(y))$ under the projection from the plane $\Pi$. Then $\operatorname{span}((\underline{x} : 0 : 0 : 0), \Pi) \subset Q(y)$ for every $\underline{x} \in L$. From Lemma 3.3 we obtain that the the proper transform of the line $L$ under the blow-up $\psi$ is contracted by $\phi$. $\square$

In the theorem below we identify curves in $\mathbb{P}_4$ with their proper transforms under the blow-up $\psi$: whenever we say a line (resp. a conic) we mean its proper transform.

**Theorem 3.6.** *There are four types of fibers $\phi^{-1}(y)$ of the map $\phi : \tilde{X}_5 \longrightarrow \mathbb{P}_3$:*

a) *union of the conic residual to the cubic $\mathcal{B} \cap \Phi^{-1}(y)$ in the planar quintic $X_5 \cap \Phi^{-1}(y)$ with the components of the cubic that satisfy the conditions of Lemma 3.4 iff $\operatorname{rank}(\mathfrak{q}(y)) \in \{5, 6, 7\}$ and $\operatorname{rank}(\mathfrak{b}(y)) = 2$ (i.e. a singularity of $Q(y)$ lies on $\Pi$),*
b) *a line in $\mathbb{P}_4$ iff $\operatorname{rank}(\mathfrak{q}(y)) = 6$ and $\operatorname{rank}(\mathfrak{b}(y)) = 3$ (equivalently $\operatorname{sing}(Q(y)) \cap \Pi = \emptyset$),*
c) *one point iff $\operatorname{rank}(\mathfrak{q}(y)) = 7$ and $\operatorname{rank}(\mathfrak{b}(y)) = 3$,*
d) *two points iff $\operatorname{rank}(\mathfrak{q}(y)) = 8$.*

*Proof.* Suppose that $\operatorname{rank}(\mathfrak{b}(y)) = 3$. Then the linear space $\operatorname{span}(\{(\underline{x} : 0 : 0 : 0) : \underline{x} \in \psi(\Phi^{-1}(y))\}, \Pi)$ is 4-dimensional and $\operatorname{sing}(Q(y)) \cap \Pi = \emptyset$. In view of Lemma 3.5, we can assume that $\operatorname{rank}(\mathfrak{q}(y)) \ge 7$ and the line $\psi(\Phi^{-1}(y)) = \{\underline{x} : \mathfrak{b}(y)\underline{x} = 0\}$ is not contained in $X_5$.
Moreover, by (22), for every point $x = (\underline{x}, x_5, x_6, x_7) \in \operatorname{span}(\{(\underline{x} : 0 : 0 : 0) : \underline{x} \in \psi(\Phi^{-1}(y))\}, \Pi)$ we have

$$x^T \mathfrak{q}(y) x = \underline{x}^T \underline{\mathfrak{q}}(y) \underline{x} \,. \tag{27}$$

Observe, that the quadratic form given by $\underline{\mathfrak{q}}(y)$ does not vanish identically on the line $\{\underline{x} : \mathfrak{b}(y)\underline{x} = 0\}$ because the latter is not contained in $X_5$. Consequently, intersection of $Q(y)$ with the linear 4-space $\mathrm{span}(\{(\underline{x} : 0 : 0 : 0) \,:\, \underline{x} \in \psi(\Phi^{-1}(y))\}, \Pi)$ consists of either one or two 3-spaces.

Lemma 3.3 implies that the fibre $\phi^{-1}(y)$ consists of a unique point iff the restriction

$$Q(y)|_{\mathrm{span}(\{(\underline{x}:0:0:0)\,:\,\underline{x}\in\psi(\Phi^{-1}(y))\},\Pi)} \tag{28}$$

is a full square.

Suppose that the fibre in question is one point. From (27) there exists a point $\underline{v} \in \mathbb{P}_5$, such that

$$\mathfrak{b}(y)\underline{v} = 0 \quad \text{and} \quad \underline{\mathfrak{q}}(y)\underline{v} = 0$$

which means that $(\underline{v} : 0 : 0 : 0) \in \mathrm{sing}(Q(y))$ and $\mathrm{rank}(\mathfrak{q}(y)) < 8$.

Assume that $\mathrm{rank}(\mathfrak{q}(y)) < 8$. Then $Q(y)$ is a cone with the unique vertex $(\underline{v} : v_5 : v_6 : v_7)$ away from the plane $\Pi$. The latter yields $\underline{v} \neq 0$. Moreover, since the tangent space to $Q(y)$ in each point contains the vertex we have $\mathfrak{b}(y)\underline{v} = 0$ and

$$(\underline{v} : v_5 : v_6 : v_7) \in \mathrm{span}(\{(\underline{x} : 0 : 0 : 0) \,:\, \underline{x} \in \psi(\Phi^{-1}(y))\}, \Pi)$$

Now $(\underline{v} : v_5 : v_6 : v_7)$ is a singularity of the restriction (28), so the polynomial $\underline{x}^T\,\underline{\mathfrak{q}}(y)\underline{x}$ has a unique double root on the line $\{\underline{x} : \mathfrak{b}(y)\underline{x} = 0\}$ and (28) is a full square.

Assume that $y \in \mathbb{P}_3$ is a point such that $\mathrm{rank}(\mathfrak{b}(y)) = 2$, and maintain the notation of the proof of Lemma 3.4. Then $y = y^{(i)}$ for an $i \in \{1, \ldots, 10\}$. By definition of the map $\phi$, the proper transform under the blow-up $\psi$ of the (possibly reducible) conic residual to (23) in the quintic (26) is always contracted by $\phi$. Moreover, a component of (23) is contracted iff it satisfies the conditions of Lemma 3.4.

Observe that rank of the quadric $Q(y^{(i)})$ does not exceed 7 because we have $\mathrm{rank}(\mathfrak{b}(y^{(i)})) = 2$. □

*Remark* 3.7. By Lemma 1.4.d there are exactly ten fibers of $\phi$ of the type a). The number of fibers of type b) will be discussed in the next section (see Cor. 4.7).

## 4. Discriminant of the web $W$

In this section we maintain the notation and the assumptions of the previous ones. In particular we assume that [A1], [A2], [A3] hold. Let $S_8$ stand for the discriminant surface of the web $W$. From now on we assume that

**[A4]:** *the discriminant surface $S_8$ has only isolated singularities .*

To simplify notation we put

$$\mathbb{I}_l := [a_{i,j}]_{i,j=0,\ldots,7}, \text{ where } a_{i,i} = 1 \text{ for } i = 1, \ldots, l \text{ and } a_{i,j} = 0 \text{ otherwise.}$$

At first we give conditions when a singularity of $S_8$ is a node:

**Lemma 4.1.** *Let $Q_0$ be a rank-7 quadric in the web $W$.*

a) *The quadric $Q_0$ is a smooth point of $S_8$ iff* $\mathrm{sing}(Q_0) \notin X_{16}$.

b) *The quadric $Q_0$ is a node of $S_8$ iff* $\mathrm{sing}(Q_0) \in X_{16}$.

*Proof.* Let $\mathfrak{q}_k =: [q_{i,j}^{(k)}]_{i,j=0,\ldots,7}$ and let $\mathcal{Q}^{(k)} := (q_{0,7}^{(k)}, \ldots, q_{6,7}^{(k)})$. After an appropriate change of coordinates we can assume that $\mathfrak{q}_0 = \mathbb{I}_7$. In particular, $\mathrm{sing}(Q_0) = \{(0 : \ldots : 0 : 1)\}$.

Let $\mathfrak{G} := [\mathfrak{g}_{i,j}]_{i,j=1,2,3}$, where $\mathfrak{g}_{i,j} := \langle \mathcal{Q}^{(i)}, \mathcal{Q}^{(j)} \rangle$ and $\langle \cdot, \cdot \rangle$ stands for the bilinear form defined by the identity matrix. By direct computation we have

$$\det(\mathfrak{q}_0 + \sum_{k=1}^{3} \mu_k \cdot \mathfrak{q}_k) = (\sum_{k=1}^{3} \mu_k \cdot q_{7,7}^{(k)}) - ((\mu_1, \mu_2, \mu_3)\ \mathfrak{G}\ (\mu_1, \mu_2, \mu_3)^T) + (\text{terms of degree} \geq 3)\,.$$

*a)* Obviously, $(1:0:0:0)$ is a smooth point of $S_8$ iff the vector $(q^{(1)}_{7,7}, q^{(2)}_{7,7}, q^{(3)}_{7,7})$ does not vanish. The latter holds iff $(1:0:0:0) \notin X_{16}$, which concludes the proof.
*b)* $(\Rightarrow)$: the implication in question results immediately from the part a).
$(\Leftarrow)$: Assume that $(q^{(1)}_{7,7}, q^{(2)}_{7,7}, q^{(3)}_{7,7}) = 0$. Then, $Q_0 = (1:0:0:0) \in \mathrm{sing}(S_8)$ is a node iff the matrix $\mathfrak{G}$ has maximal rank, i.e. $\mathcal{Q}^{(1)}$, $\mathcal{Q}^{(2)}$, $\mathcal{Q}^{(3)}$ are linearly independent. Moreover, we have $(0:\ldots:0:1) \in \mathrm{sing}(X_{16})$.

Suppose that $\mathrm{rank}(\mathfrak{G}) < 3$. Then, the last row in a matrix obtained as a non-trivial linear combination of the matrices $\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3$ vanishes, which means that the point $(0:\ldots:0:1)$ is a singularity of a quadric that belongs to $\mathrm{span}(\{Q_1, Q_2, Q_3\})$. In particular, the quadric in question does not coincide with $Q_0$. The latter is impossible by Lemma 1.4.b. Contradiction. $\square$

In the rank-6 case we have the following characterization.

**Lemma 4.2.** *Let $Q_0$ be a rank-6 quadric in the web $W$.*
a) *The quadric $Q_0$ is a node of $S_8$ iff $\mathrm{sing}(Q_0) \not\subseteq Q$ for all $Q \neq Q_0$, $Q \in W$.*
b) *$Q_0$ is an $A_m$ singularity, where $m \geq 2$, iff $\mathrm{sing}(Q_0) \cap \Pi = \emptyset$ and there exists a quadric $Q \in W$, $Q \neq Q_0$ such that $\mathrm{sing}(Q_0) \subset Q$.*
c) *The quadric $Q_0$ is a double point of the surface $S_8$.*

*Proof.* As in the proof of Lemma 4.1 we change the coordinates in such a way that $\mathfrak{q}_0 = \mathbb{I}_6$. Then, the line $\mathrm{sing}(Q_0)$ is the set of zeroes of the coordinates $x_0, \ldots, x_5$. Let $\langle \cdot, \cdot \rangle_-$ be the bilinear form on $\mathbb{C}^3$ given by the formula:

$$\langle (q^{(1)}_{6,,6}, q^{(1)}_{6,7}, q^{(1)}_{7,7}), (q^{(2)}_{6,,6}, q^{(2)}_{6,7}, q^{(2)}_{7,7}) \rangle_- := 1/2 \cdot (q^{(1)}_{6,,6} \cdot q^{(2)}_{7,7} + q^{(1)}_{7,7} \cdot q^{(2)}_{6,,6} - 2 q^{(1)}_{6,7} q^{(2)}_{6,7}) \tag{29}$$

and let $\mathfrak{H} := [\mathfrak{h}_{i,j}]_{i,j=1,2,3}$, where $\mathfrak{h}_{i,j} := \langle (q^{(i)}_{6,,6}, q^{(i)}_{6,7}, q^{(i)}_{7,7}), (q^{(j)}_{6,,6}, q^{(j)}_{6,7}, q^{(j)}_{7,7}) \rangle_-$ . By direct computation we have

$$\det(\mathfrak{q}_0 + \sum_{k=1}^{3} \mu_k \cdot \mathfrak{q}_k) = ((\mu_1, \mu_2, \mu_3)\ \mathfrak{H}\ (\mu_1, \mu_2, \mu_3)^T) + (\text{terms of degree} \geq 3)\,. \tag{30}$$

*a)* Observe that, by (30), the quadric $Q_0$ is a node of $S_8$ iff $\mathrm{rank}(\mathfrak{H}) = 3$.
$(\Rightarrow)$: Suppose that there exists a quadric $Q \neq Q_0$, $Q \in W$ such that $\mathrm{sing}(Q_0) \subset Q$. If $Q$ is given by the matrix $[q_{i,j}]_{i,j=0,\ldots,7}$, then $q_{6,6}$, $q_{6,7}$, $q_{7,7}$ vanish, which yields that $\mathrm{rank}(\mathfrak{H}) < 3$.
$(\Leftarrow)$: If $\mathrm{rank}(\mathfrak{H}) < 3$, then we can find a matrix $\mathfrak{q} = [q_{i,j}]_{i,j=0,\ldots,7}$ such that $\mathfrak{q} \in \mathrm{span}(\{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3\})$ and the entries $q_{6,6}$, $q_{6,7}$, $q_{7,7}$ vanish. The latter means that the quadric $Q$ given by $\mathfrak{q}$ contains the line $\mathrm{sing}(Q_0)$. We have $Q \neq Q_0$ because $Q_0 \notin \mathrm{span}(\{Q_1, Q_2, Q_3\})$.
*b)* By part a) we can assume that $\mathrm{sing}(Q_0) \subset Q_1$, which implies that the entries $q^{(1)}_{6,6}$, $q^{(1)}_{6,7}$, $q^{(1)}_{7,7}$ of the matrix $\mathfrak{q}_1$ vanish. Moreover, by (30), the quadric $Q_0$ is an $\mathrm{A}_m$ singularity, where $m \geq 2$, iff $\mathrm{rank}(\mathfrak{H}) = 2$ (see e.g. [9, Prop. 8.14]).
$(\Rightarrow)$: Suppose that $P \in \mathrm{sing}(Q_0) \cap \Pi$. Then $P \in \mathrm{sing}(X_{16})$ and there exists a quadric in the pencil $\mathrm{span}(\{Q_2, Q_3\})$ that meets the line $\mathrm{sing}(Q_0)$ only in the point $P$. In particular we can assume that $Q_2 \cap \mathrm{sing}(Q_0) = \{P\}$ and $P := (0:\ldots:0:1)$. The latter yields

$$q^{(2)}_{6,,6} = 1 \text{ and } q^{(2)}_{6,,7} = q^{(2)}_{7,,7} = 0.$$

Furthermore, since $P \in Q_3$ we have $q^{(3)}_{7,7} = 0$. Then

$$((\mu_1, \mu_2, \mu_3)\ \mathfrak{H}\ (\mu_1, \mu_2, \mu_3)^T) = -(q^{(3)}_{6,7})^2 \cdot \mu_3^2\,,$$

which implies that $Q_0$ is not an $\mathrm{A}_m$ singularity of the octic surface $S_8$.
$(\Leftarrow)$: By Lemma 4.2.a we have $\mathrm{rank}(\mathfrak{H}) \leq 2$, so it suffices to show that $\mathrm{rank}(\mathfrak{H}) \notin \{0, 1\}$.

Assume that $\mathrm{rank}(\mathfrak{H}) = 1$. This means that

$$\mathrm{rank}\begin{bmatrix} \mathfrak{h}_{2,2} & \mathfrak{h}_{2,3} \\ \mathfrak{h}_{3,2} & \mathfrak{h}_{3,3} \end{bmatrix} = 1\,. \tag{31}$$

Suppose that the vectors $(q^{(2)}_{6,,6}, q^{(2)}_{6,7}, q^{(2)}_{7,7})$, $(q^{(3)}_{6,,6}, q^{(3)}_{6,7}, q^{(3)}_{7,7})$ are linearly independent. By replacing $\mathfrak{q}_2$ with an appropriate linear combination of $\mathfrak{q}_2$, $\mathfrak{q}_3$ we can assume that the first column of the matrix (31) vanishes. Then, from (29) and $\mathfrak{h}_{2,2} = 0$ we obtain the equality $\mathrm{rank}([q^{(2)}_{i,j}]_{i,j=6,7}) = 1$. Performing an appropriate change of coordinates on the line $\mathrm{sing}(Q_0)$ we arrive at

$$q^{(2)}_{6,,6} = 1 \text{ and } q^{(2)}_{6,,7} = q^{(2)}_{7,,7} = 0. \tag{32}$$

Then, the equality $\mathfrak{h}_{3,2} = 0$ yields $q^{(3)}_{7,7} = 0$. The latter implies that

$$(0 : \ldots : 0 : 1) \in \mathrm{sing}(Q_0) \cap \mathrm{sing}(X_{16})\,.$$

Finally, the assumption [A1] gives $P \in \mathrm{sing}(Q_0) \cap \Pi$.
Suppose that (31) holds and the vectors $(q^{(2)}_{6,,6}, q^{(2)}_{6,7}, q^{(2)}_{7,7})$, $(q^{(3)}_{6,6}, q^{(3)}_{6,7}, q^{(3)}_{7,7})$ are linearly dependent. Then, we can assume that the entries $q^{(2)}_{6,6}$, $q^{(2)}_{6,7}$, $q^{(2)}_{7,7}$ vanish, which implies $\mathrm{sing}(Q_0) \subset Q_2$. Finally, since the line $\mathrm{sing}(Q_0)$ is contained in the quadrics $Q_1$, $Q_2$, each point in the intersection $Q_3 \cap \mathrm{sing}(Q_0)$ is a singularity of $X_{16}$. By [A1] we have $\mathrm{sing}(Q_0) \cap \Pi \neq \emptyset$.

In the same way the equality $\mathrm{rank}(\mathfrak{H}) = 0$ implies $\mathrm{sing}(Q_0) \cap \Pi \neq \emptyset$. We omit the details.
*c)* By parts a) and b) we can assume that $\mathrm{sing}(Q_0) \subset Q_1$ and $\mathrm{sing}(Q_0) \cap \Pi \neq \emptyset$. Suppose that $\mathfrak{H} = 0$. From $\mathfrak{h}_{2,2} = 0$ we obtain (32). Then $\mathfrak{h}_{3,2} = 0$ yields $q^{(3)}_{7,7} = 0$, and by $\mathfrak{h}_{3,3} = 0$ the entry $q^{(3)}_{6,6}$ vanishes. By replacing $\mathfrak{q}_3$ with $(\mathfrak{q}_3 - \mathfrak{q}_2)$ we obtain the inclusion $\mathrm{sing}(Q_0) \subset Q_3$.

To complete the proof we assume, as in Section 1 (see the proof of Remark 1.8), that the plane $\Pi$ (resp. the line $\mathrm{sing}(Q_0)$) is given by vanishing of the coordinates $x_0, \ldots, x_4$ (resp. $x_0, \ldots, x_3$ and $x_6, x_7$). Observe that the point $P = (0 : \ldots : 0 : 1 : 0 : 0) \in \mathrm{sing}(Q_0) \cap \Pi$ is a singularity of $X_{16}$. Therefore, Lemma 1.4.b yields that the quadrics $Q_1$, $Q_2$, $Q_3$ are smooth in $P$. By direct computation, there exist $v_1, \ldots, v_4 \in \mathbb{C}$ such that the intersection of the tangent spaces $\mathrm{T}_P Q_1$, $\mathrm{T}_P Q_2$, $\mathrm{T}_P Q_3$ is parametrized by the map

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \mapsto (\lambda_1 v_1, \lambda_1 v_2, \lambda_1 v_3, \lambda_1 v_4, \lambda_2, \lambda_3, \lambda_4)\,.$$

Substituting the above parametrization to (dehomogenized) $Q_0$ we see that the tangent cone $\mathrm{C}_P X_{16}$ is contained in union of two 3-planes, so the point $P \in X_{16}$ is not a node. Contradiction (see Lemma 1.6). $\square$

*Remark* 4.3. Direct computation with help of [15], gives examples of webs of quadrics such that the assumptions [A1], [A2], [A3] [A4] are fulfilled and the quadric $Q_0$ satisfies the conditions of Lemma 4.2.b. One can check that for generic choice of the quadrics one obtains an $\mathrm{A}_3$ singularity of the discriminant octic $S_8$.

To complete the description of singularities of $S_8$ we prove the following lemma.

**Lemma 4.4.** *A quadric $Q_0 \in W$ is a point of multiplicity at least 3 on $S_8$ iff* $\mathrm{rank}(\mathfrak{q}_0) = 5$.

*Proof.* ($\Rightarrow$): Lemmata 4.1, 4.2 imply that $\mathrm{rank}(Q) \leq 5$. Remark 1.8 completes the proof.
($\Leftarrow$): Assume that $\mathfrak{q}_0 = \mathbb{I}_5$ and compute the determinant $\det(\mathfrak{q}_0 + \sum_{k=1}^{3} \mu_k \cdot \mathfrak{q}_k)$. $\square$

The example below shows that the bound of Remark 1.8 is sharp, and the discriminant octic $S_8$ can have triple points.

*Example* 4.5. We define the following matrices:

$$\mathfrak{q}_0 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & -4 & 0 & -2 & 1 \\ 0 & 0 & 0 & 4 & 3 & 0 & 2 & -4 \\ 0 & 0 & -4 & 3 & 8 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 2 & -5 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \mathfrak{q}_1 := \begin{bmatrix} -4 & -4 & 2 & -1 & 0 & -1 & -1 & -3 \\ -4 & 2 & 0 & 0 & 4 & -2 & 0 & -1 \\ 2 & 0 & 0 & -1 & 2 & -2 & 4 & 2 \\ -1 & 0 & -1 & 2 & 3 & -1 & 3 & -2 \\ 0 & 4 & 2 & 3 & -4 & -2 & 0 & 1 \\ -1 & -2 & -2 & -1 & -2 & 0 & 0 & 0 \\ -1 & 0 & 4 & 3 & 0 & 0 & 0 & 0 \\ -3 & -1 & 2 & -2 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathfrak{q}_2 := \begin{bmatrix} 4 & -3 & -3 & -2 & 1 & -3 & -3 & -1 \\ -3 & -2 & -3 & -4 & 1 & 4 & 3 & 1 \\ -3 & -3 & 4 & 1 & 0 & 1 & 1 & 1 \\ -2 & -4 & 1 & 2 & -2 & 0 & 1 & 4 \\ 1 & 1 & 0 & -2 & 4 & -1 & 0 & -1 \\ -3 & 4 & 1 & 0 & -1 & 0 & 0 & 0 \\ -3 & 3 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 4 & -1 & 0 & 0 & 0 \end{bmatrix} \quad \mathfrak{q}_3 := \begin{bmatrix} 4 & -1 & 2 & 2 & -2 & -1 & -2 & 0 \\ -1 & 2 & 2 & -3 & -1 & -4 & -2 & 4 \\ 2 & 2 & -2 & -1 & 1 & 3 & 2 & -1 \\ 2 & -3 & -1 & -2 & 0 & 1 & 3 & -2 \\ -2 & -1 & 1 & 0 & -4 & 4 & 1 & -1 \\ -1 & -4 & 3 & 1 & 4 & 0 & 0 & 0 \\ -2 & -2 & 2 & 3 & 1 & 0 & 0 & 0 \\ 0 & 4 & -1 & -2 & -1 & 0 & 0 & 0 \end{bmatrix}$$

By direct computation with help of [15], the intersection in $\mathbb{P}_7$ of the quadrics defined by the above matrices satisfies the assumptions [A1], ..., [A4]. As one can easily see, we have $\mathrm{rank}(\mathfrak{q}_0) = 5$.

We put $\pi_2 : X_8 \to W$ to denote the double cover of the web $W$ branched along the discriminant surface $S_8$. We have the following theorem (compare [24, Thm 3.1]).

**Theorem 4.6.** *Assume that [A1], ..., [A4] hold.*

a) *There exists a (small) resolution $\hat{\phi} : \tilde{X}_5 \to X_8$ of singularities of the double octic $X_8$ such that the following diagram commutes:*

$$\begin{tikzcd} \tilde{X}_5 \arrow[rr, "\phi"] \arrow[dr, "\hat{\phi}"'] & & \mathbb{P}_3 \\ & X_8 \arrow[ur, "\pi_2"'] & \end{tikzcd}$$

b) *Let $\pi$ be the map induced by the projection from the plane $\Pi$ (see (9)) and let $\sigma$ (resp. $\psi$) be the blow up defined by (6) (resp. (20)). Then the composition*

$$X_{16} \overset{\sigma^{-1}}{\dashrightarrow} \tilde{X}_{16} \overset{\pi}{\longrightarrow} X_5 \overset{\psi^{-1}}{\dashrightarrow} \tilde{X}_5 \overset{\hat{\phi}}{\longrightarrow} X_8$$

*is a birational map between the base locus of the web $W$ and its double cover branched along the discriminant surface $S_8$. In particular, the base locus $X_{16}$ and the discriminant double octic $X_8$ are birational to the quintic 3-fold $X_5$ (see (12)) that contains Bordiga sextic.*

*Proof.* a) Consider Stein factorization of the map $\phi : \tilde{X}_5 \to \mathbb{P}_3$:

$$\phi = \hat{\phi} \circ \phi'$$

where $\phi'$ is finite and $\hat{\phi}$ has connected fibers. By Cor. 3.2 the map $\phi'$ is a (ramified) double cover of $\mathbb{P}_3$. Thm 3.6 and the assumption [A4] imply the equality $\phi' = \pi_2$. Then the map $\hat{\phi} : \tilde{X}_5 \longrightarrow X_8$ is birational (see e.g. [8, p. 11]). Thm 3.6 implies that the set of 1-dimensional fibers of the latter map coincides with $\hat{\phi}^{-1}(\mathrm{sing}(X_8))$. This completes the proof.
b) We have just shown that the map $\hat{\phi}$ is birational. The claim follows from Lemma 2.5.a. $\square$

In the case of the double sextic defined by a net of quadrics that contain a (fixed) line the discriminant curve has only nodes as singularities (see [7, Thm 3.3]).

In the corollary below we discuss the singularities of the discriminant surface $S_8$.

**Corollary 4.7.** *Assume that [A1], …, [A4] hold.*

a) *The equality* $\sum_{P \in \mathrm{sing}(X_8)}(\mu(P, X_8) + 1) = 188$ *holds, where* $\mu(P, X_8)$ *stands for the Milnor number of* $X_8$ *in the point* $P$.

b) *A quadric* $Q_0 \in W$ *is a singularity of* $S_8$ *of the type given in the first column of the table below iff it satisfies the conditions listed in the other column*

| *Type of singularity* | *Conditions* | | |
|---|---|---|---|
| | $\mathrm{rank}(\mathfrak{q}_0)$ | | |
| *smooth point* | 7 | $\mathrm{sing}(Q_0) \cap X_{16} = \emptyset$ | |
| $A_1$ | 7 | $\mathrm{sing}(Q_0) \cap X_{16} \neq \emptyset$ | |
| | 6 | | $\{Q \in W \,:\, Q \neq Q_0, \mathrm{sing}(Q_0) \subset Q\} = \emptyset$ |
| $A_m$, $m \geq 3$, $m$ *odd* | 6 | $\mathrm{sing}(Q_0) \cap \Pi = \emptyset$ | $\{Q \in W \,:\, Q \neq Q_0, \mathrm{sing}(Q_0) \subset Q\} \neq \emptyset$ |
| *double point of corank* 2 | 6 | $\mathrm{sing}(Q_0) \cap \Pi \neq \emptyset$ | $\{Q \in W \,:\, Q \neq Q_0, \mathrm{sing}(Q_0) \subset Q\} \neq \emptyset$ |
| $k$*-fold point,* $k \geq 3$ | 5 | | |

*Proof.* a) To compute the sum of Milnor numbers of singularities of $X_8$ we compare topological Euler numbers of $\tilde{X}_5$ and $X_8$. By the assumption [A3] and Lemma 2.7 we have $e(\tilde{X}_5) = -108$. On the other hand, by Chern class argument the Euler number of a smooth octic in $\mathbb{P}_3$ is 304, so [10, Cor. 5.4.4] implies $e(X_8) = -296 + \sum_{P \in \mathrm{sing}(X_8)} \mu(P, S_8)$. Observe that in our set-up the equality $\mu(P, S_8) = \mu(P, X_8)$ holds. From Thm 4.6.a we get

$$-108 + \#(\mathrm{sing}(X_8) = -296 + \sum_{P \in \mathrm{sing}(X_8)} \mu(P, X_8)\,. \tag{33}$$

that yields the claim.

b) By Thm 4.6.b and [28, Cor. 1.16] the octic $S_8$ has no $\mathrm{A}_m$ points with $m$ even. The claim follows now directly from Lemmata 4.1, 4.2, and Lemma 4.4. □

*Remark* 4.8. Under the assumptions [A1], …, [A4] the following inequality holds

$$\#\{P \in \mathrm{sing}(S_8) \,:\, P \text{ is not an } \mathrm{A}_m \text{ point, where } m \geq 1\} \leq 10.$$

*Proof.* By Lemmata 4.1, 4.2 each double point $Q_0 \in \mathrm{sing}(S_8)$ that is not an $\mathrm{A}_m$ singularity is a singular quadric and its singular locus meets the plane $\Pi$. The same holds for rank-5 quadrics in the web $W$ (see Thm 3.6). Therefore, the inequality results from Remark 3.7. □

*Final remarks:* a) According to [21, Thm 4.1] the normal bundle a smooth rational curve that is contracted on a 3-fold is one of the following: $(\mathcal{O}_{\mathbb{P}_1}(-1) \oplus \mathcal{O}_{\mathbb{P}_1}(-1))$, $(\mathcal{O}_{\mathbb{P}_1}(-2) \oplus \mathcal{O}_{\mathbb{P}_1})$, $(\mathcal{O}_{\mathbb{P}_1}(-3) \oplus \mathcal{O}_{\mathbb{P}_1}(1))$. Remark 4.3 and Ex. 4.5 show that all such bundles can come up in our set-up. For the conditions imposed on the equation of a (smooth) 3-fold quintic in $\mathbb{P}_4$ by the normal bundle of a contracted curve the reader should consult [20, App. A, B].

b) Assume that all singularities of $S_8$ are A-D-E points. By [3, Thm 1.1] the Hodge diamond of any small Kähler resolution of the double octic $X_8$ coincides with the one given in Lemma 1.7. In view of [29, Cor. 5.1] and [ibid., Prop. 6.1], the latter implies that the assumptions [A1], …, [A4] determine position of singularities of $S_8$ with respect to sections of $\mathcal{O}_{\mathbb{P}_3}(8)$ (compare [24, Prop. 2.13]).

c) In Thm 3.6 we describe components of $\Phi^{-1}(y)$ when $\mathrm{rank}(\mathfrak{b}(y)) = 2$. Since all singularities of $X_8$ admit a small resolution, [25, Thm 5.5] can be applied to obtain a more precise description of such fibers. We omit details because of lack of space.

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